Morse Inequalities and Bergman Kernels

Inaugural-Dissertation
zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

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voraussichtlicher Tag
der mündlichen Prüfung: 19 April 2013

Abstract

This thesis consists of two parts. In part I, we prove equivariant Morse inequalities via Bismut-Lebeau's analytic localization techniques. As an application, we obtain Morse inequalities on compact manifold with nonempty boundary by applying equivariant Morse inequalities to the doubling manifold. In part II, we calculate the second coefficient of the asymptotic expansion of the Bergman kernel of the Hodge-Dolbeault operator associated to high powers of a Hermitian line bundle with non-degenerate curvature, using the method of formal power series developed by Ma and Marinescu.

Keywords Equivariant Morse inequalities, Analytic localization techniques, Hodge-Dolbeault operator, Bergman kernel, Asymptotic expansion.

Kurzzusammenfassung

Diese Arbeit besteht aus zwei Teilen. Im ersten Teil werden wir die äquivarianten Morse-Ungleichungen mit Hilfe der analytischen Lokalisierungstechniken von Bismut-Lebeau beweisen. Als eine Anwendung erhalten wir die Morse-Ungleichungen auf kompakten Mannigfaltigkeiten mit nicht-leerem Rand, in dem wir die Morse-Ungleichungen auf die doubling Mannigfaltigkeit anwenden.

Im zweiten Teil berechnen wir den zweiten Koeffizienten der Entwicklung des Bergman-Kerns des Hodge-Dolbeault-Operators, der mit dem Komplex der Differentialformen mit Werten in grossen Potenzen eines hermiteschen Geradenbündels mit nicht-degenerierter Krümmung assoziiert wird. Dabei benutzen wir die Methode der formalen Potenzreihen, die von Ma und Marinescu entwickelt wurde.

Stichwörter äquivariante Morse-Ungleichungen, analytische Lokalisierungstechniken, Hodge-Dolbeault-Operator, Bergmanscher Kern, asymptotische Entwicklung.

Acknowledgement

First, I am deeply grateful to my advisor Professor Dr. George Marinescu for his advices. I am grateful for his consistent encouragement, especially during the calculations in the second part of the Thesis. I am also indebted to Professor Xiaonan Ma in Institut de mathématiques de Jussieu for enlightening discussions on the topics in my Thesis while I was visiting Paris.

Moreover, I wish to thank the former and current members of the graduate school who created a warm and friendly working atmosphere.

Last but not least, I would like to thank my brother and my parents for their consistent support during my years in Germany.

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1 Introduction

A function f on a smooth manifold is called Morse function if all its critical points are isolated and non-degenerate. The weak Morse inequalities state that the Betti number β_j of a smooth closed manifold bounds up by the number of critical points of a Morse function with the same index j. It is known that Morse functions always exist on a smooth compact manifold. Moreover, the space of Morse functions is dense in the space of smooth functions on smooth compact manifold.

The standard Morse inequalities were first proved by topologists using the change of topology of the half-spaces $M_a = \{x \in M | \; f(x) \leqslant a\}$ associated with a Morse function f on a closed manifold M [34]. Thom and Smale observed the importance of the space of trajectories for the gradient flow of f. Under certain transversality condition of these trajectories (Morse-Smale condition) one can form a chain complex, called Thom-Smale complex, whose chains are generated by the critical points of f and whose boundary operator is defined by counting gradient flow lines (with sign). The cohomology of the Thom-Smale complex is isomorphic to the singular homology of the manifold with integer coefficients. This isomorphism immediately yields the standard Morse inequalities. Witten introduced a deformation of the de Rham complex by using a Morse function and indicated that the Thom-Smale complex is isomorphic to a subcomplex of the deformed de-Rham complex, consisting of eigenspaces of small eigenvalues of the associated deformed de-Rham operator.

In Zhang's book [56], especially [56, Chapter 6], we find an analytic proof of Witten's intuitions, by using the analytic localization techniques developed by Bismut-Lebeau ([6, §8-9]).

In his influential work [52], Witten sketched analytic proofs of the degenerate Morse inequalities of Bott [8] for Morse functions whose critical submanifolds are non-degenerate in the sense of Bott. Rigorous proofs were given by Bismut [4], by using heat kernel methods, and later by Helffer and Sjöstrand [22], by means of semiclassical analysis. Braverman and Farber [12] provided another proof using the Witten deformation techniques suggested by Bismut [4].

Concerning the standard Morse inequalities (i.e., for Morse functions with isolated critical points), an analytic proof is given by Zhang [56, Chapter 5], in the spirit of the analytic localization techniques developed by Bismut-Lebeau [6, §8-9]. Moreover, [56, Chapter 6] contains a complete proof of the isomorphism between the Thom-Smale complex and the Witten instanton complex. Following the ideas in [56], we give here a proof of degenerate Morse inequalities by similar techniques.

Let us mention the related papers [12, 13, 18]. In [12, 13], Braverman, Farber and Silantyev used Witten deformation techniques to study the Novikov number associated to closed differential 1-forms non-degenerate in the sense of Bott and Kirwan, respectively.

In this way, they obtained Novikov-type inequalities associated to a closed differential 1-form. When the closed differential form is exact, these inequalities turn to Morse inequalities.

In [18], Feng and Guo establish Nivokov's type inequalities associated to vector fields instead of closed differential forms under a natural assumption on the zero-set of the vector field.

In this thesis, we give a proof of the equivariant Morse-Bott inequalities along the lines of [56] (cf. [6, §8-9]). Note that our theorem does not follow from the Morse inequalities of Braverman-Faber [12]. Moreover, it is difficult to adapt Bismut's heat kernel approach [4] in the equivariant situation. On the other hand, such equivariant Morse-Bott inequalities are probably folklore among topologists. Compared to [18], where Bismut-Lebeau's analytic localization techniques are applied along the lines of [56], we can choose the geometrical data near the singular points as simple as possible, due to the equivariant Morse's Lemma [51]. As an application, we get degenerate Morse inequalities for manifolds with nonempty boundary by passing to the doubling manifold. Thus, we extend the result from [54] to the most general situation.

Let us now turn to our second topic of this Thesis.

The study of the asymptotic expansion of Bergman kernels has attracted much attention recently. The existence of the asymptotic expansion of the Bergman kernel of Dirac operator acting on high power tensors of positive line bundle over compact complex manifold was established by Catlin [14] and Zelditch [55]. Tian [50], followed by Ruan [39] and Lu [26], computed many terms of the asymptotic expansion on the diagonal via Tian's method of peak solutions.

Using Bismut-Lebeau's analytic localization techniques, Dai, Liu and Ma [17] established the full off diagonal asymptotic expansion of the Bergman kernel of the Spin^c Dirac operator associated to high powers of a Hermitian line bundle with positive curvature in the general context of symplectic manifold. Moreover, they calculated the first coefficient of the expansion in the case of Kähler manifolds. Later, Ma and Marinescu [31] studied the expansion of generalized Bergman kernels and developed a method of formal power series to computer the coefficients. By the same method, Ma and Marinescu [29, Theorem 2.1] compute the first coefficient of the asymptotic expansion of the Bergman kernel of the Spin^c Dirac operator acting on high tensor powers of line bundles with positive curvature in the case of symplectic manifolds.

Here we consider the Hodge-Dolbeault operator, which is a modified Dirac operator, associated to a high power of a Hermitian line bundle with non-degenerate curvature over compact complex manifold. For the non-degenerate curvature case, R. Berman and J. Sjöstrand [3] studied the asymptotic expansion for Bergman kernels for high powers of complex line bundles. Ma and Marinescu [29] obtained the expansion [29, Theorem 1.7] of the Bergman kernel of the Spin^c Dirac operator in non-degenerate curvature case and they computed the coefficient [29, Theorem 2.1] in the case of positive curvature assumption. Our work in this thesis is a continuation of their work [29]. We compute the second coefficient of asymptotic expansion of the Hodge-Dolbeault operator via the method in [29, 31]. Compared to [29, 31], the main feature here is that we do our calculations without the positive curvature assumption.

The organization of this Thesis is as follows. We will prove the Equivariant Morse inequalities and the degenerate Morse inequalities for manifold with boundary in Chapter 2. Chapter 3 is devoted to the calculation of the second coefficient of the asymptotic expansion of the Bergman kernel of the Hodge-Dolbeault operator associated to high powers of a Hermitian line bundle with non-degenerate curvature. In Appendix 4, we give a complete proof of the known fact that, given a Morse function, there exists a metric on M such that the corresponding gradient vector field satisfies Morse-Smale conditions [35], [45]. Appendix 4 plays a crucial role in the construction of Thom-Smale-Witten complex.

2 Equivariant Morse inequalities and applications

The contents of this Chapter is as follows. In Section 2.1, we state the first main result of this Thesis. In Section 2.2, we describe briefly the Standard Morse inequalities. In Section 2.3, we prove the degenerate Morse inequalities along the lines of [56] (cf. [6, §8-9]). A key point is to compare the relations between the kernel spaces of Dirac operator on critical manifolds and the eigenspaces associated to small eigenvalues of the deformed de-Rham operator of M following [52].

In Section 2.4, we give a proof of equivariant Morse inequalities (Theorem 2.1), motivated by the idea in [56]. For a finite group G, we first construct a $G-$isomorphism between the two spaces considered in Section 2.3, i.e., the kernel of Dirac operator on critical manifolds and the eigenspaces associated to small eigenvalues of the deformed de-Rham operator of M. Using this G-isomorphism, we immediately get the equivariant Morse inequalities. When $G = \{1\}$, the equivariant Morse inequalities reduce to the degenerate Morse inequalities.

In Section 2.5, we establish the general Morse inequalities (Theorem 2.2) for manifolds with possibly non-empty boundary. We first consider the special case that $f|_{\partial M} = 0$, where we divide our proof into three special cases. When N_+ or N_- (cf. Section 2.1) is empty and we reduce the equivariant case by using the $\mathbb{Z}_2$ action to the doubling manifold of M. In the general case, we need to consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action and paste the manifold twice along the boundary N_+, N_- respectively. Combining the result in the special case and the degenerate Morse inequalities, we get Theorem 2.2.

2.1 Main result

Let M be a smooth m-dimensional closed and connected Riemannian manifold and G be a finite group acting on M as diffeomorphism. Let $f : M \to \mathbb{R}$ be a smooth G-invariant Morse-Bott function [8]. That is, the critical points of f form a union of disjoint connected submanifolds $B_1, B_2, \cdots, B_r$ and if $x \in B_i$, the Hessian of f is non-degenerate on any subspace of T_xM which intersects T_xB_i transversally. For $1 \leqslant i \leqslant r$, let n_i be the dimension of the submanifold B_i and n_i^- be the index of the Hessian of f on B_i.

Using equivariant Morse-Bott Lemma [51], we embed each critical submanifold B_i in a tubular neighborhood $(h, N_i^- \oplus N_i^+)$ so that

$$f \circ h(Z^-, Z^+) = c - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \tag{2.1.1}$$

where c denote the constant value of $f(B_i)$ and the rank of N_i^- is n_i^-, while that of N_i^+ is $m - n_i - n_i^-$. Let $o(N_i^-)$ denote the orientation bundle of N_i^-. We call n_i^- the index of B_i in M.

In the sequel, we will often omit the subscript i in B_i, n_i, n_i^-, i.e., n denotes the dimension of the critical submanifold B and n^- is the index.

Let W_1, W_2 be two finite-dimensional real representations of G. Let $\mathrm{Hom}_G(W_1, W_2)$ denote the set of all linear $G-$morphism between W_1 and W_2. If E_1, E_2 are two finite-dimensional real representations of G, then we denote

$$E_1 \leqslant E_2 \tag{2.1.2}$$

in the representation ring $R(G)$ if for any irreducible representation V of G, the multiplicity of V in E_1 is smaller than the multiplicity of V in E_2, equivalently,

$$\dim\,\mathrm{Hom}_G(V, E_1) \leqslant \dim\,\mathrm{Hom}_G(V, E_2). \tag{2.1.3}$$

The first main result in the Thesis is as follows.

Theorem 2.1. *Let M be a smooth m-dimensional closed and connected manifold, and let G be a finite group acting smoothly on M. Let $f : M \to \mathbb{R}$ be a smooth G-invariant Morse-Bott function. Then we have for $k = 0, 1, \ldots, m$,*

$$\sum_{j=0}^{k}(-1)^{k-j}H^j(M) \leqslant \sum_{i=1}^{r}\sum_{j=n_i^-}^{k}(-1)^{k-j}H^{j-n_i^-}(B_i, o(N_i^-)) \tag{2.1.4}$$

in the sense of (2.1.2). When $k = m$, the equality holds,

$$\sum_{j=0}^{m}(-1)^{m-j}H^j(M) = \sum_{i=1}^{r}\sum_{j=n_i^-}^{m}(-1)^{m-j}H^{j-n_i^-}(B_i, o(N_i^-)). \tag{2.1.5}$$

Let us explain Theorem 2.1 in some details.

Set

$$F_j = \bigoplus_{i=1}^{r} H^{j-n_i^-}(B_i, o(N_i^-)). \tag{2.1.6}$$

F_j is a finite-dimensional real vector space. We denote by q_j the dimension of F_j,

$$q_j = \sum_{i=1}^{r}\dim\,H^{j-n_i^-}(B_i, o(N_i^-)). \tag{2.1.7}$$

Let $\{V^\alpha\}_1^{l_0}$ be the irreducible representations of G, which is finite. As G-representation, we have the following decomposition:

$$F_j = \bigoplus_{\alpha=1}^{l_0}\mathrm{Hom}_G(V^\alpha, F_j) \otimes V^\alpha, \tag{2.1.8}$$

and

$$H^j(M) = \bigoplus_{\alpha=1}^{l_0} \mathrm{Hom}_G(V^\alpha, H^j(M)) \otimes V^\alpha. \tag{2.1.9}$$

For integers $j = 0, 1, \cdots, m, \alpha = 1, \cdots, l_0$, set

$$d_j^\alpha = \dim\ \mathrm{Hom}_G(V^\alpha, F_j), \tag{2.1.10}$$

and

$$b_j^\alpha = \dim\ \mathrm{Hom}_G(V^\alpha, H^j(M)). \tag{2.1.11}$$

Then (2.1.4) is equivalent to say for $k = 0, 1, \cdots, m, \alpha = 1, \cdots, l_0$,

$$\sum_{j=0}^{k}(-1)^{k-j} b_j^\alpha \leqslant \sum_{j=0}^{k}(-1)^{k-j} d_j^\alpha, \tag{2.1.12}$$

and (2.1.5) is equivalent to:

$$\sum_{j=0}^{m}(-1)^{m-j} b_j^\alpha = \sum_{j=0}^{m}(-1)^{m-j} d_j^\alpha. \tag{2.1.13}$$

From the above equivariant Morse inequalities (2.1.12) and (2.1.13), we will obtain the Morse inequalities for manifold with boundary.

Let M be an m-dimensional smooth oriented, connected manifold with nonempty boundary ∂M. Let $f : M \to \mathbb{R}$ be a smooth function such that it is a Morse-Bott function in the interior of M. Let $f|_{\partial M}$ be restriction of f to the boundary. We assume that the following condition holds. Let $\partial M = N_+ \sqcup N_-$ be a disjoint union of closed manifolds such that $f(y, u) = \frac{1}{2}u^2 + f_+(y)$ in a collar neighborhood of $N_+ \times [0, 1)$, while $f(y, u) = -\frac{1}{2}u^2 + f_-(y)$ in a collar neighborhood of $N_- \times [0, 1)$, here f_+ (resp. f_-) is a Morse-Bott function on N_+ (resp. N_-). That is, $f|_{\partial M}$ is also a Morse-Bott function.

Let $N_+ = N_{a+} \sqcup N_{r+}$ and $N_- = N_{a-} \sqcup N_{r-}$ be disjoint union of closed manifolds. The subscripts "a" and "r" refer respectively to absolute and relative boundary condition. Set $N_a = N_{a+} \sqcup N_{a-}, N_r = N_{r+} \sqcup N_{r-}$. We assume that in the collar neighborhood $\partial M \times [0, 1)$, Riemannian metric is assumed to take the product form $g^{TM} = g^{T(\partial M)} \oplus d^2u$, where $g^{T(\partial M)}$ is a Riemannian metric on ∂M.

Let $\{B_i\}_{i=1}^r$ (resp. $\{S_{+,i}\}_{i=1}^{t_+}$, resp. $\{S_{-,i}\}_{i=1}^{t_-}$) be the critical submanifolds of f in the interior of M (resp. of f_+ on N_+, resp. of f_- on N_-).

Set

$$S_{a+,i} = S_{+,i} \cap N_a, \quad S_{r-,i} = S_{-,i} \cap N_r. \tag{2.1.14}$$

Let $o(N_i^-)$ denote the orientation bundle over B_i as before. To simplify our notation, we denote by $o(S_{a+,i})$ (resp. $o(S_{r-,i})$) the corresponding bundle on $S_{a+,i}$ (resp. $o(S_{r-,i})$) and by $n_{a+,i}^-$ (resp. $n_{r-,i}^-$) be its index in N_a (resp. N_r).

Set

$$F_{a+,j} = \bigoplus_{i=1}^{t_+} H^{j-n^-_{a+,i}}(S_{a+,i}, o(S_{a+,i})), \quad q_{a+,j} = \dim\ F_{a+,j};$$
$$F_{r-,j} = \bigoplus_{i=1}^{t_-} H^{j-n^-_{r-,i}}(S_{r-,i}, o(S_{r-,i})), \quad q_{r-,j} = \dim\ F_{r-,j}. \tag{2.1.15}$$

Theorem 2.2. *The following inequalities hold for* $k = 0, 1, \cdots, m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, N_r) \leqslant \sum_{j=0}^{k}(-1)^{k-j}\mu_j, \tag{2.1.16}$$

where

$$\beta_j(M, N_r) = \dim\ H^j(M, N_r), \mu_j = q_j + q_{a+,j} + q_{r-,j-1}. \tag{2.1.17}$$

The equality holds for k=m.

When $f|_{\partial M} = 0$ and the critical points of f are isolated and non-degenerate, Theorem 2.2 reduces to [54, Theorem 1].

2.2 Standard Morse inequalities

In this Section, we briefly review some basic materials about standard Morse inequalities. See [34] for more details.

Let M be a smooth m-dimensional closed and connected manifold. Let f be a smooth real valued function on M. A point $p \in M$ is called a critical point of f if the induced map $f_* : T_pM \to T_{f(p)}(\mathbb{R})$ is zero, here T_pM denotes the tangent space of M at the point p. If we choose a local coordinate system $(x_1, \cdots, x_m)$ in a neighborhood U of p, the critical point p is called non-degenerate if and only if the (Hessian) matrix $[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)]$ is non-singular. We say that f is a Morse function if every critical point of f is non-degenerate.

Let f be a Morse function on M. It is clear that every non-degenerate critical point of f is isolated. Thus, the number of critical points of f are finite. Let $d^2 f$ denote the Hessian of f. Then index of $d^2 f$ is defined to be the dimension of the maximum subspace of T_pM on which $d^2 f$ is negative definite. The index of $d^2 f$ on T_pM will be referred to simply as the index of f at p.

The following Lemma, known as Morse Lemma [34, Lemma 2.2], is important in the study of properties of topology of manifold via Morse functions.

Lemma 2.3. *Let p be a non-degenerate critical points of f. Then there is a local coordinate system $(x_1, \cdots, x_m)$ in a neighborhood U of p with $x_j(p) = 0$ for all j and such that the identity*

$$f = f(p) - \frac{x_1^2}{2} - \cdots - \frac{x_\lambda^2}{2} + \frac{x_{\lambda+1}^2}{2} + \cdots + \frac{x_m^2}{2} \tag{2.2.1}$$

holds through U, where λ is the index of f at p.

For $j = 0, 1, \cdots, m$, let b_j be the j-th Betti number, i.e., $b_j = \dim H^j(M, \mathbb{R})$ and c_j denote the number of critical points with index j.

The following inequalities are known as strong Morse inequalities [34, §5].

Theorem 2.4. *For any $k = 0, 1, \cdots, m$,*

$$\sum_{j=0}^{k} (-1)^{k-j} b_j \leqslant \sum_{j=0}^{k} (-1)^{k-j} c_j. \tag{2.2.2}$$

When $k = m$, the equality holds:

$$\sum_{j=0}^{m} (-1)^{m-j} b_j = \sum_{j=0}^{m} (-1)^{m-j} c_j. \tag{2.2.3}$$

Adding (2.2.2) for $k = j$ and (2.2.2) for $k = j - 1$, one gets

$$b_j \leqslant c_j, \quad j = 0, \cdots, m. \tag{2.2.4}$$

The inequalities (2.2.4) are known as weak Morse inequalities.

2.3 Degenerate Morse Inequalities

In this Section, we will prove Theorem 2.1 when G is trivial.

2.3.1 Statement of main result when G is trivial

Let M be a smooth m-dimensional closed and connected Riemannian manifold and $f : M \to \mathbb{R}$ be a Morse-Bott function. Then Theorem 2.1 takes the following form [4, Theorem 2.14].

Theorem 2.5. *For $k = 0, 1, \cdots, m$*

$$\sum_{j=0}^{k} (-1)^{k-j} b_j \leqslant \sum_{j=0}^{k} (-1)^{k-j} q_j. \tag{2.3.1}$$

For $k = m$, the equality holds,

$$\sum_{j=0}^{m} (-1)^{m-j} b_j = \sum_{j=0}^{m} (-1)^{m-j} q_j. \tag{2.3.2}$$

If all critical manifolds B_i are isolated, i.e., f is a Morse function, then q_j is precisely the number of critical points with critical index j, (2.3.1) and (2.3.2) turn to the standard Morse inequalities (2.2.2) and (2.2.3).

Since (2.3.1) and (2.3.2) are purely topological result, we may and will choose a special Riemannian metric on the manifold. This will greatly simply the proof.

2.3.2 Some calculations on Euclidian space

In this subsection, we will perform some calculations on Euclidian space. The result of this subsection will be applied to the fibres of the normal bundle N to B in M.

Let V be an l-dimensional real vector space endowed with an Euclidean scalar product. Let V^+, V^- be two subspaces such that $V = V^- \oplus V^+$ and $\dim V^- = n^-$. Let $e_1, \cdots, e_l$ be an orthonormal basis on V such that V^- is spanned by $e_1, \cdots, e_{n^-}$. Let $f \in C^\infty(V, \mathbb{R})$ defined by:

$$f(Z) = f(0) - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \tag{2.3.3}$$

where $Z^- = (Z_1, \cdots, Z_{n^-}), Z^+ = (Z_{n^-+1}, \cdots, Z_l), (Z^-, Z^+)$ denote the coordinate functions on V corresponding to the decomposition $V = V^- \oplus V^+$.

Let

$$Z = \sum_{\alpha=1}^{l} Z_\alpha e_\alpha \tag{2.3.4}$$

be the tautological vector field on V. There is a natural Euclidean scalar product on ΛV^*. Let $dv_V(Z)$ be the volume form on V. Let Γ be the set of the square integrable sections of ΛV^* over V. For $\alpha, \beta \in \Gamma$, set

$$\big\langle \alpha, \beta \big\rangle = \int_V \langle \alpha, \beta \rangle_{\Lambda V^*} dv_V(Z). \tag{2.3.5}$$

Let d be the usual differential operator acting on the smooth section of ΛV^* and δ be the formal adjoint of d with respect to the Euclidean product (2.3.5).

Let $C(V)$ be the Clifford algebra of V, i.e., the algebra generated over $\mathbb{R}$ by $e \in V$ and the communication relations $ee' + e'e = -2\langle e, e' \rangle$ for $e, e' \in V$. Let $c(e), \widehat{c}(e)$ be the Clifford action on ΛV^* defined by

$$c(e) = e^* \wedge -i_e, \ \widehat{c}(e) = e^* \wedge +i_e, \tag{2.3.6}$$

where $e^* \wedge$ and i_e are the standard notation for exterior and interior multiplication and e^* denotes the dual of e by the Euclidean scalar product on V. Then ΛV^* is a Clifford module.

If $X, Y \in V$, one has

$$\begin{aligned} c(X)c(Y) + c(Y)c(X) &= -2\langle X, Y \rangle, \\ \widehat{c}(X)\widehat{c}(Y) + \widehat{c}(Y)\widehat{c}(X) &= 2\langle X, Y \rangle, \\ c(X)\widehat{c}(Y) + \widehat{c}(Y)c(X) &= 0. \end{aligned} \tag{2.3.7}$$

We denote by v the gradient of f with respect to the given Euclidean scalar product, then

$$v(Z) = -\sum_{\alpha=1}^{n^-} Z_\alpha e_\alpha + \sum_{\alpha=n^-+1}^{l} Z_\alpha e_\alpha. \tag{2.3.8}$$

Let Δ be the standard Laplacian on V, i.e.,

$$\Delta = -\sum_{\alpha=1}^{l}(\frac{\partial}{\partial Z_\alpha})^2. \tag{2.3.9}$$

Set

$$d_T = e^{-Tf} d \cdot e^{Tf},\ \delta_T = e^{-Tf}\delta \cdot e^{Tf},\ D_{T,v} = d_T + \delta_T = d + \delta + T\widehat{c}(v).$$

Let $e^1, \cdots, e^l$ be the dual basis of $e_1, \cdots, e_l$. Then we have the following result [52],[56, Proposition 4.9].

Proposition 2.6. *The kernel of $D^2_{T,v}$ is one dimension and is spanned by*

$$\beta = e^{-\frac{T|Z|^2}{2}} e^1 \wedge \cdots \wedge e^{n^-}. \tag{2.3.10}$$

Moreover, all nonzero eigenvalues of $D^2_{T,v}$ are $\geqslant 2T$.

Proof. For $e \in V$, let ∇_e be the differential operator along the vector e.
It is easy to calculate the square of $D_{T,v}$,

$$\begin{aligned} D^2_{T,v} &= \Delta + T^2|Z|^2 + T\sum_{\alpha=1}^{l} c(e_\alpha)\widehat{c}(\nabla_{e_\alpha} v) \\ &= (\Delta + T^2|Z|^2 - Tl) + T\sum_{\alpha=1}^{n^-}[1 - c(e_\alpha)\widehat{c}(e_\alpha)] + T\sum_{\alpha=n^-+1}^{l}[1 + c(e_\alpha)\widehat{c}(e_\alpha)] \\ &= (\Delta + T^2|Z|^2 - Tl) + 2T(\sum_{\alpha=1}^{n^-} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=n^-+1}^{l} e^\alpha \wedge i_{e_\alpha}). \end{aligned} \tag{2.3.11}$$

The operator

$$\mathcal{L}_T = \Delta + T^2|Z|^2 - Tl \tag{2.3.12}$$

is the harmonic oscillator operator on V. By [19, Theorem 1.5.1], [30, Appendix E], we know that $\mathcal{L}_T$ is nonnegative elliptic operator with the kernel of dimension one and generated by $e^{-\frac{T|Z|^2}{2}}$. Moreover, the nonzero eigenvalues of $\mathcal{L}_T$ are all greater than $2T$. It is also easy to verify that the linear operator

$$\sum_{\alpha=1}^{n^-} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=n^-+1}^{l} e^\alpha \wedge i_{e_\alpha} \tag{2.3.13}$$

is nonnegative with the kernel being one dimensional and generated by

$$e^1 \wedge \cdots \wedge e^{n^-}. \tag{2.3.14}$$

The proof of Proposition 2.6 is complete. □

2.3.3 Local analysis near critical manifolds

In this subsection, we give the geometric setting for this Section.

By the generalized Morse Lemma [23], we know that B possesses a tubular neighborhood (h, N) such that:
(1) N is a vector bundle over B, which is endowed with the scalar product g^N. Moreover N, which has rank $m-n$, splits into two orthogonal subbundles $N = N^- \oplus N^+$, where the rank of N^- is n^- and the rank of N^+ is n^+.
(2) h embeds N into M. Moreover there is an open neighborhood $\mathcal{B}$ of B in N such that if $Z = (Z^-, Z^+) \in \mathcal{B}$, then

$$f(h(Z)) = c - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \tag{2.3.15}$$

where c denotes the constant values $f(B)$.

In the sequel, we will identify N and $h(N)$. Let π be the projection $N \to B$.

Let g^{TB} be a Riemannian metric on TB and ∇^{TB} be the Levi-Civita connection on TB associated to g^{TB}. As Euclidean bundles, (N^-, g^{N^-}) (resp. (N^+, g^{N^+})) can be endowed with Euclidean connections ∇^{N^-}(resp. ∇^{N+}). We then have a natural Euclidean connection ∇^N on N, i.e.,

$$\nabla^N = \nabla^{N^-} \oplus \nabla^{N^+}. \tag{2.3.16}$$

The Euclidean connection ∇^N on N induces a splitting $TN = T^H N \oplus T^V N$ of the tangent space of the total space N [2, Proposition 1.20], where $T^H N$ is the horizontal part of TN with respect to the Euclidean connection ∇^N and $T^V N$ is isomorphic to $\pi^* N$. If $X \in TB$, let X^H denote the horizontal lift of X in $T^H N$ such that $X^H \in T^H N, \pi_* X^H = X$.

If $y \in N$, then π_* identifies $T^H_y N$ with $T_{\pi(y)} B$. Moreover, $T^V_y N$ and N can be naturally identified. In this way, $T^H_y N$ and $T^V_y N$ are both endowed with scalar product induced by g^{TB} and g^N, i.e., we get a metric $g^{TN} = \pi^* g^{TB} \oplus g^N$ on $TN = T^H N \oplus T^V N$. Here we still denote g^N as the induced metric on $T^V N$. Let ∇^{TN} be the Levi-Civita connection on N associated to the Riemannian metric g^{TN}.

Let $TN|_B$ be the restriction of the tangent bundle TN to B. Recall that N is identified with the bundle orthogonal to TB in $TN|_B, TN|_B = TB \oplus N$. Let $\nabla^{TN|_B}$ be the restriction of ∇^{TN} to $TN|_B$.

Lemma 2.7. *The following identity holds:*

$$\nabla^{TN|_B} = \nabla^{TB} \oplus \nabla^N. \tag{2.3.17}$$

Proof. Let P^{TB} (resp. P^N) be the orthogonal projection from $TN|_B$ to TB (resp. N). We need to prove that B is totally geodesic submanifold of N with respect to the metric g^{TN} and that

$$\nabla^{TB} = P^{TB} \nabla^{TN|_B} P^{TB}, \ \nabla^N = P^N \nabla^{TN|_B} P^N. \tag{2.3.18}$$

The first equality in (2.3.18) is easily proved by the uniqueness of the Levi-Civita connection on B.

Fix $y_0 \in B$. Let $e_1(y_0), \cdots, e_l(y_0)$ be an orthonormal basis of N_{y_0}. Let $\mathbf{y}$ denote the tautological vector field in $T_{y_0}B$. Let $\tau^N_{t\mathbf{y}}$ be the parallel transport of section of N along the curve $t \longrightarrow t\mathbf{y}, t \in [0,1]$ with respect to the connection ∇^N. Set

$$e_\alpha(\mathbf{y}) = \tau^N_{\mathbf{y}}\big(e_\alpha(y_0)\big). \tag{2.3.19}$$

This gives a trivialization of N over a neighborhood U of y_0 in B. Let $w = (w_{\alpha\beta}), w_{\alpha\beta} \in \Omega^1(U)$ be the connection one-form of N with respect to the trivialization:

$$N|_U \simeq U \times \mathbb{R}^l,\ Z = \sum_{\alpha=1}^{l} Z_\alpha e_\alpha \longmapsto (\pi(Z), Z_1, \cdots, Z_l), \tag{2.3.20}$$

and

$$\nabla^N = d + w,\ w_{\mathbf{y}}(\cdot)Z = \sum_{\alpha,\beta=1}^{l} w_{\mathbf{y},\alpha\beta}(\cdot) e_\alpha Z_\beta. \tag{2.3.21}$$

Then $w_{\mathbf{y},\alpha\beta}$ is antisymmetric over α and β. Moreover, as vector space, $N_{\mathbf{y}}$ is naturally identified to its tangent space, thus

$$e_\alpha(\mathbf{y}) = \frac{\partial}{\partial Z_\alpha}(\mathbf{y}). \tag{2.3.22}$$

Let R^N the curvature operator of Euclidean connection ∇^N, i.e., $R^N = (\nabla^N)^2$. From [2, Proposition 1.18],

$$w_{\mathbf{y}}(\cdot) = \frac{1}{2} R^N_{y_0}(\mathbf{y}, \cdot) + O(|\mathbf{y}|^2). \tag{2.3.23}$$

In particular, $w_{0,\alpha\beta} = 0$.

The tautological vector field $\mathbf{y}$ on $T_{y_0}B$ may also be regarded as a point of U. Sometimes we also use y to denote the point $\mathbf{y}$ in U.

For $y \in U, Z \in N_y$, we have the decomposition:

$$T_{(y,Z)}N = T^H_{(y,Z)}N + T^V_{(y,Z)}N, \tag{2.3.24}$$

where $T^V N$ is the vertical bundle over B and the horizontal part $T^H N$ may be expressed as [2, Proposition 1.20]:

$$T^H_{(y,Z)}N = \big\{(X, -w(X)Z) | X \in T_y B\big\}. \tag{2.3.25}$$

By definition, the metric $\pi^* g^{TB}$ on $T^H N$ is given by:

$$(\pi^* g^{TB})_{(y,Z)}(W_1, W_2) = g^{TB}_y(\pi_* W_1, \pi_* W_2). \tag{2.3.26}$$

By definition of T^HN, the identification between TN and $T^HN \oplus T^VN$ is given by

$$\begin{aligned} TN \simeq TU \times \mathbb{R}^n &\to T^HN \oplus T^VN \\ (Y, W) &\mapsto (Y - w(Y)Z, W + w(Y)Z). \end{aligned} \tag{2.3.27}$$

Then g^{TN} can be written out explicitly.

$$\begin{aligned} & g^{TN}_{(y,Z)}((Y_1, W_1),\ (Y_2, W_2)) \\ &= (\pi^* g^{TB})_{(y,Z)}(Y_1 - w(Y_1)Z, Y_2 - w(Y_2)Z) + g^N_y(W_1 + w(Y_1)Z, W_2 + w(Y_2)Z) \\ &= g^{TB}_y(Y_1, Y_2) + g^N_y(W_1 + w(Y_1)Z, W_2 + w(Y_2)Z). \end{aligned} \tag{2.3.28}$$

Let $f_1(0), \cdots, f_n(0)$ be an orthonormal basis of $T_{y_0}B$. We denote by y_j the coordinate system on $T_{y_0}B = \mathbb{R}^n$ such that $\mathbf{y} = \sum_{j=1}^n y_j f_j(0)$ holds. We denote by $\tau^B_{t\mathbf{y}}$ the parallel transport of $f_j(y_0)$ along the curve $t \longrightarrow t\mathbf{y},\ t \in [0,1]$ with respect to the Levi-Civita connection ∇^{TB}. Set

$$f_j(\mathbf{y}) = \tau^B_{\mathbf{y}}(f_j(y_0)).$$

Then $f_j(\mathbf{y})$ is a local orthonormal frame of B and

$$\left(\nabla^{TB}_{\cdot} f_j\right)_{y_0} = 0. \tag{2.3.29}$$

Then $\{f_j\}_{j=1}^n$ and $\{e_\alpha\}_{\alpha=1}^l$ form an orthonormal frame of $TN|_B$.

To prove that B is totally geodesic in N with respect to the metric g^{TN}, it suffice to show

$$\left\langle \nabla^{TN}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j}, \frac{\partial}{\partial Z_\alpha} \right\rangle_{y_0} = 0. \tag{2.3.30}$$

By the formula [2, (1.18)] of the Levi-Civita connection ∇^{TN}, we have

$$2\left\langle \nabla^{TN}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j}, \frac{\partial}{\partial Z_\alpha} \right\rangle = \frac{\partial}{\partial y_i}\left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial Z_\alpha} \right\rangle + \frac{\partial}{\partial y_j}\left\langle \frac{\partial}{\partial Z_\alpha}, \frac{\partial}{\partial y_i} \right\rangle - \frac{\partial}{\partial Z_\alpha}\left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right\rangle. \tag{2.3.31}$$

It is clear that

$$\left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial Z_\alpha} \right\rangle_{\mathbf{y}} = \left\langle f_j, e_\alpha \right\rangle_{\mathbf{y}} = 0. \tag{2.3.32}$$

Therefore

$$\begin{aligned} 2\left\langle \nabla^{TN}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j}, \frac{\partial}{\partial Z_\alpha} \right\rangle_{y_0} &= -\frac{\partial}{\partial Z_\alpha}\left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right\rangle_{y_0} \\ &= -\frac{\partial}{\partial Z_\alpha}\Big|_{Z=0}\Big[g^{TB}\left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right\rangle + g^N\left\langle w(\frac{\partial}{\partial y_i})Z, w(\frac{\partial}{\partial y_j})Z \right\rangle\Big] \\ &= 0. \end{aligned} \tag{2.3.33}$$

The last equation in (2.3.33) is due to the fact that the term $g^N_{y_0}\left\langle w(\frac{\partial}{\partial y_i})Z, w(\frac{\partial}{\partial y_j})Z\right\rangle$ is a quadratic form of Z.

The second formula of (2.3.18) follows from

$$\left\langle \nabla^{TN}_{\frac{\partial}{\partial y_i}} e_\alpha, e_\beta \right\rangle_{y_0} = \left\langle \nabla^{N}_{\frac{\partial}{\partial y_i}} e_\alpha, e_\beta \right\rangle_{y_0}. \tag{2.3.34}$$

We prove (2.3.34) as follows.

The right hand side of (2.3.34) is

$$\langle \nabla^N_{f_j} e_\alpha, e_\beta \rangle_{y_0} = w_{y_0,\beta\alpha}(f_j) = 0. \tag{2.3.35}$$

For the left side of (2.3.34), we have

$$\begin{aligned} 2\left\langle \nabla^{TN}_{f_j} e_\alpha, e_\beta \right\rangle =& \left\langle [f_j, e_\alpha], e_\beta \right\rangle - \left\langle [e_\alpha, e_\beta], f_j \right\rangle + \left\langle [e_\beta, f_j], e_\alpha \right\rangle \\ &+ f_j\left\langle e_\alpha, e_\beta \right\rangle + e_\alpha\left\langle e_\beta, f_j \right\rangle - e_\beta\left\langle f_j, e_\alpha \right\rangle. \end{aligned} \tag{2.3.36}$$

At the point y_0, we find that

$$f_j\left\langle e_\alpha, e_\beta \right\rangle_{y_0} = 0. \tag{2.3.37}$$

From (2.3.28), we have

$$\left\langle e_\beta, f_j \right\rangle_{(y,Z)} = \left\langle e_\beta, w(f_j)Z \right\rangle_y. \tag{2.3.38}$$

Then

$$e_\alpha\left\langle e_\beta, f_j \right\rangle_{y_0} = e_\alpha\left\langle e_\beta, w(f_j)Z \right\rangle_{y_0} = w_{y_0,\beta\alpha}(f_j) = 0. \tag{2.3.39}$$

Similarly,

$$e_\beta\left\langle e_\alpha, f_j \right\rangle_{y_0} = 0. \tag{2.3.40}$$

It is clear that

$$\left\langle [e_\alpha, e_\beta], f_j \right\rangle_{y_0} = 0.$$

From (2.3.22), we find that

$$[f_j, e_\alpha]_{\mathbf{y}} = 0, \quad [f_j, e_\beta]_{\mathbf{y}} = 0. \tag{2.3.41}$$

Thus the left side of (2.3.34) equals to 0. The proof of Lemma 2.7 is complete. □

2.3.4 Proof of Theorem 2.5

Let g^{TM} be a Riemannian metric on the manifold M which coincides with g^{TN} in a neighborhood of B via the embedding h. (Using partition of unity argument, it is always possible.)

Let $o(TM)$ be the orientation line bundle on M and let dv_M be the density (or Riemannian volume form) on M. Note that we do not assume that M is oriented; thus $dv_M \in C^\infty(M, \Lambda^m(T^*M) \otimes o(TM))$ [2, Page 29], [10, Page 88]. Let $\Omega(M)$ be the set of smooth sections of $\Lambda(T^*M)$ on M. For $f, g \in \Omega(M)$, set

$$\langle f, g \rangle = \int_M \langle f, g \rangle (x) dv_M(x). \tag{2.3.42}$$

Let D^M be the classic Dirac operator on M, i.e.,

$$D^M = d + \delta, \tag{2.3.43}$$

where d the exterior differential operator and δ is the adjoint of d with respect to the metric (2.3.42). Let ∇f be the gradient vector field of f with respect to the Riemannian metric g^{TM} on M. Set

$$\begin{aligned} d_T &= e^{-Tf} d \cdot e^{Tf}, \quad \delta_T = e^{Tf} \delta \cdot e^{-Tf}, \\ D_T &= d_{Tf} + \delta_{Tf} = D^M + T\widehat{c}(\nabla f). \end{aligned} \tag{2.3.44}$$

Following the argument after [56, Proposition 5.5], one easily gets degenerate Morse inequalities (2.3.1) and (2.3.2) if the following Proposition holds.

Proposition 2.8. *There exist $C_0 > 0, T_0 > 0$ such that when $T > T_0$, the number of eigenvalues in $[0, C_0)$ of $D_T^2|_{\Omega^j(M)}$ equals to q_j.*

Proof of Theorem 2.5. Let $\mathrm{F}_{T,j}^{C_0}$ be the q_j-dimensional vector space generated by the eigenspaces of $D_T^2|_{\Omega^j(M)}$ associated to the eigenvalues lying in $[0,\ C_0)$. Since $d_T D_T = D_T d_T$, $d_T(F_{T,j}^{C_0}) \subset F_{T,j+1}^{C_0}$. Then we have the following complex:

$$(\mathrm{F}_{T,\cdot}^{C_0}, d_T) : 0 \longrightarrow \mathrm{F}_{T,0}^{C_0} \longrightarrow \mathrm{F}_{T,1}^{C_0} \longrightarrow \cdots \to \mathrm{F}_{T,m}^{C_0} \longrightarrow 0.$$

By Hodge Theorem, the j-th cohomology group of the above complex is isomorphic to $\mathrm{Ker}\big(D_T^2|_{\Omega^j(M)}\big)$, which is again by Hodge Theorem isomorphic to the j-th cohomology group of the complex $(\Omega(M), d_T)$. Then (2.3.1) and (2.3.2) follows from standard algebraic techniques ([30, Lemma 3.2.12]). This completes the proof of Theorem 2.5. $\square$

The rest of this Section is left to prove Proposition 2.8.

2.3.5 Local expansion of D_T near the critical submanifold B

In this subsection, we study the local behavior of the twisted operator D_T near the sub-manifold B.

We first introduce a coordinate system on M near B. If $y \in B, Z \in N_y$, let $y_t = \exp_y(tZ), t \in \mathbb{R}$ be the geodesic in M with $y_0 = y, \frac{dy_t}{dt}|_{t=0} = Z$. For $\varepsilon > 0$, set $\mathcal{B}_\varepsilon = \{(y,Z) \in N; y \in B, |Z| < \varepsilon\}$. Here and after, $|Z|$ always stands for $|Z|_{g^N_y}$. Since B and M are compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the map $(y,Z) \in N \to \exp_y Z \in M$ is a diffeomorphism from $\mathcal{B}_\varepsilon$ onto a tubular neighborhood $\mathcal{U}_\varepsilon$ of B in M. From now on, we identify $\mathcal{B}_\varepsilon$ with $\mathcal{U}_\varepsilon$ and use the notation $x = (y,Z)$ instead of $x = \exp_y Z$. Finally, we identify $y \in B$ with $(y,0) \in N$.

Take $\alpha > 0$. Let $\mathbf{E}$ (resp. $\mathbf{E}_\alpha$) be the set of smooth sections of $\pi^*(\Lambda(T^*M)|_B)$ on the total space of N (resp. of $\pi^*(\Lambda(T^*M)|_B)$ over $\mathcal{B}_\alpha$).

The symbols dv_B and dv_N are understood in the same manner as dv_M. Let $f_1, \cdots, f_n$ (resp. $e_1, \cdots, e_l$) be the orthonormal basis of T_yB (resp. N_y) as in Lemma 2.7. From (2.3.28), we know that $e_1, \cdots, e_l$ are also orthonormal basis at the points (y,Z) on the total space N. It is clear that

$$dv_N(y,Z) = dv_B(y)dv_{N_y}(Z). \tag{2.3.45}$$

For $f, g \in \mathbf{E}$ have compact support, set

$$\langle f, g\rangle = \int_B \Big(\int_{N_y} \langle f, g\rangle(y,Z)dv_{N_y}(Z)\Big)dv_B(y). \tag{2.3.46}$$

If $f \in \mathbf{E}$ has compact support in $\mathcal{B}_{\varepsilon_0}$, we will identify f with an element of E which has compact support in $\mathcal{U}_{\varepsilon_0}$. This identification is unitary with respect to the Euclidean product (2.3.42) and (2.3.46).

Let ∇^{TM} be the Levi-Civita connection on TM. Then there exists a natural connection on $\Lambda(T^*M)$ on $\Lambda(T^*M)$ induced by ∇^{TM}, which we denote by $\nabla^{\Lambda(T^*M)}$. Let $\nabla^{\Lambda(T^*M)|_B}$ be the restriction of $\nabla^{\Lambda(T^*M)}$ to $\Lambda(T^*M)|_B$. The connection $\nabla^{\Lambda(T^*M)|_B}$ on $\Lambda(T^*M)|_B$ can be lift to a connection on the bundle $\pi^*(\Lambda(T^*M)|_B)$, which we denote by $\pi^*(\nabla^{\Lambda(T^*M)|_B})$.

Definition 2.9. Let D^H, D^N be the operators acting on $\mathbf{E}$:

$$\begin{aligned} D^H &= \sum_{j=1}^n c(f_j)(\pi^*\nabla^{\Lambda(T^*M)|_B})_{f_j^H}, \\ D^N &= \sum_{\alpha=1}^l c(e_\alpha)(\pi^*\nabla^{\Lambda(T^*M)|_B})_{e_\alpha}. \end{aligned} \tag{2.3.47}$$

We now prove that D^H is self-adjoint with respect to metric (2.3.46). For $s_1, s_2 \in C^\infty(N, \pi^*(\Lambda T^*M|_B))$ and one of them has compact support, then

$$\begin{aligned}\big\langle D^H s_1, s_2\big\rangle =&\sum_{j=1}^n \int_B\int_{N_y} \Big\langle c(f_j)\big(\pi^*\nabla^{\Lambda(T^*M)|_B}\big)_{f_j^H} s_1, s_2\Big\rangle(y,Z)dv_{N_y}(Z)dv_B(y)\\ =&\sum_{j=1}^n \int_B\int_{N_y} \Big\langle \big(\pi^*\nabla^{\Lambda(T^*M)|_B}\big)_{f_j^H}\big(c(f_j)s_1\big), s_2\Big\rangle(y,Z)dv_{N_y}(Z)dv_B(y)\\ &-\sum_{j=1}^n \int_B\int_{N_y} \Big\langle c\Big(\big(\pi^*\nabla^{\Lambda(T^*M)|_B}\big)_{f_j^H} f_j\Big)s_1, s_2\Big\rangle(y,Z)dv_{N_y}(Z)dv_B(y)\\ =&\sum_{j=1}^n \int_B\int_{N_y} \Big[f_j^H\Big\langle c(f_j)s_1, s_2\Big\rangle - \Big\langle c(f_j)s_1, \big(\pi^*\nabla^{\Lambda(T^*M)|_B}\big)_{f_j^H} s_2\big)\Big\rangle\Big]dv_{N_y}(Z)dv_B(y)\\ &-\sum_{j=1}^n \int_B\int_{N_y} \Big\langle c\Big(\big(\pi^*\nabla^{(T^*M)|_B}\big)_{f_j} f_j\Big)s_1, s_2\Big\rangle(y,Z)dv_{N_y}(Z)dv_B(y)\\ =&\big\langle s_1, D^H s_2\big\rangle + I + II,\end{aligned} \tag{2.3.48}$$

where

$$I = -\sum_{j=1}^n \int_B\int_{N_y} \big(w(f_j)Z\big)\Big\langle c(f_j)s_1, s_2\Big\rangle dv_{N_y}(Z)dv_B(y),$$

and

$$II = \sum_{j=1}^n \int_B\int_{N_y} \Big[f_j\Big\langle c(f_j)s_1, s_2\Big\rangle - \Big\langle c\big(\nabla^{TB}_{f_j} f_j\big)s_1, s_2\Big\rangle\Big]dv_{N_y}(Z)dv_B(y).$$

From (2.3.21) and integration by parts, we find that

$$\begin{aligned}I =& -\sum_{j=1}^n\sum_{\alpha,\beta=1}^l \int_B\int_{N_y} Z_\alpha w_{y,\beta\alpha}(f_j)\frac{\partial}{\partial Z_\beta}\Big(\big\langle c(f_j)s_1, s_2\big\rangle\Big)dv_{N_y}(Z)dv_B(y)\\ =&\sum_{j=1}^n\sum_{\beta=1}^l \int_B\int_{N_y} w_{y,\beta\beta}(f_j)\langle c(f_j)s_1, s_2\rangle dv_{N_y}(Z)dv_B(y)\\ =&0.\end{aligned} \tag{2.3.49}$$

For the term II, we set

$$\vartheta(Y)(y) = \int_{N_y} \Big\langle c(Y)s_1, s_2\Big\rangle(y,Z)dv_{N_y}(Z), \ \text{for } Y\in C^\infty(B,TB). \tag{2.3.50}$$

Then $\vartheta \in C^\infty(B, T^*B)$ and

$$\begin{aligned}\mathrm{Tr}(\nabla\vartheta) &= \sum_{j=1}^n \Big[f_j\big(\vartheta(f_j)\big) - \vartheta\big(\nabla^{TB}_{f_j} f_j\big)\Big] \\ &= \sum_{j=1}^n \int_{N_y} \Big[f_j \Big\langle c(f_j)s_1, s_2\Big\rangle - \Big\langle c\big(\nabla^{TB}_{f_j} f_j\big)s_1, s_2\Big\rangle\Big] dv_{N_y}(Z).\end{aligned} \tag{2.3.51}$$

From [2, Proposition 2.7], we deduce that

$$II = \int_B \mathrm{Tr}\big(\nabla\vartheta\big)(y) dv_B(y) = 0. \tag{2.3.52}$$

Combining (2.3.48), (2.3.49) and (2.3.52), one immediately gets

$$\big\langle D^H s_1, s_2\big\rangle = \big\langle s_1, D^H s_2\big\rangle.$$

Similarly D^N is also self-adjoint. Moreover, D^N is formally self-adjoint along the fibres of N, i.e., for $s_1, s_2 \in \mathbf{E}$ with compact supports,

$$\int_{N_y} \langle D^N s_1, s_2\rangle(y,Z) dv_{N_y}(Z) = \int_{N_y} \langle s_1, D^N s_2\rangle(y,Z) dv_{N_y}(Z). \tag{2.3.53}$$

The proof of (2.3.53) is just by integration by parts.

Using identification $(\Lambda(T^*N))|_{(y,Z)}$ with $(\Lambda T^*N)_y$ by transport parallel along the geodesic $t \to (y, tZ), t \in [0,1]$ with respect to the connection $\nabla^{\Lambda(T^*N)}$, we can now consider the connection $\nabla^{\Lambda(T^*M)}$ as a Euclidean connection on $\pi^*(\Lambda(T^*M)|_B)$ over $\mathcal{B}_\varepsilon$.

For $T \geqslant 1$, let Q_T be a first order differential operator acting on $\mathbf{E}_{\varepsilon_0}$. Then Q_T can be written in the form

$$Q_T = \sum_{j=1}^n a_j(T,y,Z)\pi^*(\nabla^{\Lambda(T^*M)|_B})_{f_j^H} + \sum_{\alpha=1}^l b_\alpha(T,y,Z)\pi^*(\nabla^{\Lambda(T^*M)|_B})_{e_\alpha} + c(T,y,Z),$$

where $a_j(T,y,Z,), b_\alpha(T,y,Z), c(T,y,Z)$ are endomorphisms of $\pi^*(\Lambda(T^*M)|_B)$ which depend smoothly on (y,Z).

Assume there exist constants $C>0$ such that for any $T \geqslant 1$, $(y,Z) \in \mathcal{B}_{\varepsilon_0}$, then

$$\begin{aligned}|a_j(T,y,Z)| &\leqslant C|Z|^2, 1 \leqslant j \leqslant n;\\ |b_\alpha(T,y,Z)| &\leqslant C|Z|^2, 1 \leqslant \alpha \leqslant l;\\ |c(T,y,Z)| &\leqslant C(|Z| + T|Z|^4).\end{aligned} \tag{2.3.54}$$

We will then use the notation

$$Q_T = O(|Z|^2\partial^N + |Z|^2\partial^H + |Z| + T|Z|^4). \tag{2.3.55}$$

In (2.3.55), ∂^H and ∂^N represent horizontal and vertical differential operators respectively.

Recall that the vector field v is defined as in (2.3.8). Set

$$D_T^N = D^N + T\widehat{c}(v). \tag{2.3.56}$$

For the twisted operator D_T, we have the following Theorem which is an analogue of [18, Lemma 2.3] , [6, Theorem 8.18].

Theorem 2.10. *As $T \to +\infty$, we have*

$$D_T = D_T^N + D^H + O(|Z|^2\partial^N + |Z|^2\partial^H + |Z| + T|Z|^4). \tag{2.3.57}$$

Proof. The proof is easier than those of [18, Lemma 2.3], [6, Theorem 8.18] because B is now totally geodesic submanifold of the total manifold N.

We now use freely the notations in Section 2.4 as the local geometry near B is described in detail therein.

For $y \in U$, we denote by Z the tautological vector field in N_y, i.e., $Z = \sum_{\alpha=1}^{l} Z_\alpha e_\alpha$. If $(y, Z) \in \mathcal{B}_\varepsilon, X \in T_yN$, let $\tilde{X}$ be the parallel transport of X with respect to the connection ∇^{TM} along the geodesic $t \to (y, tZ),\ t \in [0,1]$, i.e.,

$$(\nabla_Z^{TM}\tilde{X})(y, Z) = 0. \tag{2.3.58}$$

Since

$$\left[\nabla_Z^{TM}, c(\tilde{X})\right] = c(\nabla_Z^{TM}\tilde{X}) = 0, \tag{2.3.59}$$

Then

$$c(\tilde{X})(y, Z) = c(X)(y). \tag{2.3.60}$$

By (2.3.58), we find that $\tilde{e}_\alpha(y, Z) = e_\alpha(y)$. From (2.3.43) and (2.3.60), we have

$$D^M = \sum_{j=1}^{n} c(f_j)\nabla_{\tilde{f}_j}^{TM} + \sum_{\alpha=1}^{l} c(e_\alpha)\nabla_{e_\alpha}^{TM}. \tag{2.3.61}$$

For $1 \leqslant j \leqslant n$, set

$$\tilde{f}_j(y, Z) = f_j(y) + \sum_{k=1}^{n}\sum_{\alpha=1}^{l} c_{kj}^{\alpha}(y)Z_\alpha f_k(y) + \sum_{\beta=1}^{l}\sum_{\alpha=1}^{l} c_{\beta j}^{\alpha}(y)Z_\alpha e_\beta(y) + O(|Z|^2), \tag{2.3.62}$$

From (2.3.58), we have

$$0 = \nabla_Z^{TM} f_j + \sum_{k=1}^{n}\sum_{\alpha=1}^{l} c_{kj}^{\alpha}(y)Z_\alpha f_k + \sum_{\beta=1}^{l}\sum_{\alpha=1}^{l} c_{\beta j}^{\alpha}(y)Z_\alpha e_\beta + O(|Z|^2). \tag{2.3.63}$$

From (2.3.41), We also find that

$$\nabla_Z^{TM} f_i = \sum_{\alpha=1}^{l} Z_\alpha \nabla_{e_\alpha}^{TM} f_j = \sum_{\alpha=1}^{l} Z_\alpha \nabla_{f_j}^{TM} e_\alpha. \tag{2.3.64}$$

By Lemma 2.7, at the point of y,

$$\nabla^{TM}_{f_j} e_\alpha = \nabla^{N}_{f_j} e_\alpha.$$

Then (2.3.63) becomes

$$0 = \sum_{\alpha=1}^{l} Z_\alpha (\nabla^N_{f_j} e_\alpha)_y + \sum_{k=1}^{n}\sum_{\alpha=1}^{l} c^\alpha_{kj}(y) Z_\alpha f_k + \sum_{\beta=1}^{l}\sum_{\alpha=1}^{l} c^\alpha_{\beta j}(y) Z_\alpha e_\beta + O(|Z|^2). \tag{2.3.65}$$

As $\nabla^N_{f_j} e_\alpha$ has no component of f_k, from (2.3.65) it is clear that

$$c^\alpha_{kj}(y) = 0, \qquad \text{for} \ \ 1 \leqslant k \leqslant n; \tag{2.3.66}$$

and

$$c^\alpha_{\beta j}(y) = -\langle \nabla^N_{f_j} e_\alpha, e_\beta \rangle_y, \quad \text{for} \ \ 1 \leqslant \beta \leqslant l. \tag{2.3.67}$$

Thus,

$$\tilde{f}_j(y, Z) = f_j - \sum_{\beta=1}^{l}\sum_{\alpha=1}^{l} \langle \nabla^N_{f_j} e_\alpha, e_\beta \rangle_y Z_\alpha e_\beta + O(|Z|^2). \tag{2.3.68}$$

By [2, Proposition 1.20], we know that

$$\tilde{f}_j(y, Z) = f_j - \sum_{\alpha=1}^{l} (\nabla^N_{f_j} e_\alpha)_y Z_\alpha + O(|Z|^2) = f^H_j(y, Z) + O(|Z|^2). \tag{2.3.69}$$

Set

$$\Gamma = \nabla^{TM} - \pi^* \nabla^{TM|_B}. \tag{2.3.70}$$

From (2.3.61), (2.3.69) and (2.3.70), we get

$$\begin{aligned} D^M =& \sum_{j=1}^{n} c(f_j)(\pi^*\nabla^{TM|_B})_{f^H_j} + \sum_{j=1}^{n} c(f_j)\Gamma(\tilde{f}_j) + \sum_{\alpha=1}^{l} c(e_\alpha)\Gamma(e_\alpha) \\ &+ \sum_{\alpha=1}^{l} c(e_\alpha)(\pi^*\nabla^{TM|_B})_{e_\alpha} + O(|Z|^2\partial^H + |Z|^2\partial^N). \end{aligned} \tag{2.3.71}$$

From (2.3.17), we deduce that

$$\Gamma_y = 0. \tag{2.3.72}$$

Thus from (2.3.71) and (2.3.72), we find

$$D^M = D^H + D^N + O(|Z|^2\partial^H + |Z|^2\partial^N + |Z|). \tag{2.3.73}$$

Set

$$v_j(y,Z) = (\tilde{f}_j f)(y,Z), \quad v_\alpha(y,Z) = (e_\alpha f)(y,Z). \tag{2.3.74}$$

As $\{e_\alpha, \tilde{f}_j\}$ is an orthonormal frame of the total space N, then

$$\nabla f(y,Z) = \sum_{j=1}^{n} v_j(y,Z)\tilde{f}_j + \sum_{\alpha=1}^{l} v_\alpha(y,Z)e_\alpha, \tag{2.3.75}$$

Lemma 2.11. *For* $1 \leqslant j \leqslant n$,

$$v_j(y,Z) = O(|Z|^4). \tag{2.3.76}$$

Proof of Lemma 2.11. By (2.3.15) and (2.3.68),

$$\begin{aligned} v_j(y,Z) = (\tilde{f}_j f)(y,Z) &= \tilde{f}_j\Big[f(y) - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}\Big] \\ &= \Big[f_j - w(f_j)Z\Big]\Big[-\frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}\Big] + O(|Z|^4) \\ &= -\sum_{\alpha,\beta=1}^{l} Z_\alpha w_{y,\beta\alpha}(f_j)e_\beta\Big[-\frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}\Big] + O(|Z|^4). \end{aligned} \tag{2.3.77}$$

Thus

$$v_j(y,Z) = \sum_{\alpha=1}^{l}\sum_{\beta=1}^{n^-} w_{y,\beta\alpha}(f_j)Z_\alpha Z_\beta - \sum_{\alpha=1}^{l}\sum_{\beta=n^-+1}^{l} w_{y,\beta\alpha}(f_j)Z_\alpha Z_\beta + O(|Z|^4). \tag{2.3.78}$$

It is clear that $w_{y,\beta\alpha}$ is anti-symmetric in α and β. From (2.3.16), we know that $w_{y,\beta\alpha} = 0$, if $1 \leqslant \alpha \leqslant n^-, n^- + 1 \leqslant \beta \leqslant l$. Then

$$v_j(y,Z) = \sum_{\alpha,\beta=1}^{n^-} w_{y,\beta\alpha}(f_j)Z_\alpha Z_\beta - \sum_{\alpha,\beta=n^-+1}^{l} w_{y,\beta\alpha}(f_j)Z_\alpha Z_\beta + O(|Z|^4) = O(|Z|^4). \tag{2.3.79}$$

□

Now we continue the proof of Theorem 2.10.
From (2.3.15),

$$v_\alpha(y,Z) = \begin{cases} -Z_\alpha & \text{if } 1 \leqslant \alpha \leqslant n^-, \\ Z_\alpha & \text{if } n^- + 1 \leqslant \alpha \leqslant l. \end{cases} \tag{2.3.80}$$

Combining (2.3.8), (2.3.76) and (2.3.80), we get

$$\nabla f(y,Z) = -\sum_{\alpha=1}^{n^-} Z_\alpha e_\alpha + \sum_{\alpha=n^-+1}^{l} Z_\alpha e_\alpha + O(|Z|^4) = v + O(|Z|^4). \tag{2.3.81}$$

The proof of the Theorem 2.10 is complete. □

Note that D_T^N is actually an elliptic operator acting fibrewise on $\pi^*\Lambda(N^*)$.

Theorem 2.12. *For any $y \in B$, the restriction of $(D_T^N)^2$ on $C^\infty(N_y, \Lambda(N_y^*))$ is nonnegative with the kernel generated by*

$$\beta_y = e^{-\frac{T|Z|^2}{2}}\theta_y, \tag{2.3.82}$$

where θ_y is the volume form of N_y^-. Moreover, all the nonzero eigenvalues of $(D_T^N)^2$ are $\geqslant 2T$.

Proof. Let Δ^N be the positive Laplacian along the fibres of N.

From (2.3.47) and (2.3.56), it is clear that on $C^\infty\big(N, \pi^*(\Lambda(T^*M)|_B)\big)$,

$$(D_T^N)^2 = -\sum_{\alpha=1}^{l}(\pi^*\nabla_{e_\alpha}^{TM|_B})^2 + T^2|v|^2 + T\sum_{\alpha=1}^{l} c(e_\alpha)\widehat{c}(\pi^*\nabla_{e_\alpha}^{TM|_B}v). \tag{2.3.83}$$

From (2.3.80), we obtain

$$(D_T^N)^2 = \Delta^N + T^2|Z|^2 - T\sum_{\alpha=1}^{n^-} c(e_\alpha)\widehat{c}(e_\alpha) + T\sum_{\alpha=n^-+1}^{l} c(e_\alpha)\widehat{c}(e_\alpha). \tag{2.3.84}$$

For $Z \in N_y$, $\big(\pi^*(\Lambda(T^*M)|_B)\big)(y,Z) = \pi^*(\Lambda(T_y^*B)) \otimes \Lambda N_y^*$. One gets the results of Theorem 2.12 from Proposition 2.6. □

2.3.6 Some estimates on the $D_{T,j}$'s as $T \to +\infty$

In this subsection, we will give a decomposition of D_T as $\sum_{j=1}^4 D_{T,j}$ and establish some estimates about $D_{T,j}$ as $T \to +\infty$ by using Bismut-Lebeau analytic localization techniques [6].

Recall that $o(N^-)$ is the orientation bundle of the bundle N^-. We denote by $\det\big((N^-)^*\big)$ the determinant line bundle of $(N^-)^*$. The connection ∇^{N^-} on N^- induces naturally an Euclidean connection $\nabla^{\det((N^-)^*)}$ on $\det(N^-)^*$. Let $\Phi : \det\big((N^-)^*\big) \to o(N^-)$ denote the canonical isomorphism over B. Let $\nabla^{o(N^-)}$ be the Euclidean connection on $o(N^-)$ induced by $\nabla^{\det((N^-)^*)}$ via the canonical isomorphism $\Phi : \det((N^-)^*) \to o(N^-)$. Since there exists canonical metric on $o(N^-)$ (which is independent of the trivialization of the bundle $o(N^-)$), we could find canonical Euclidean connection on $o(N^-)$, which is just the exterior differential operator d. However, given the canonical metric on $o(N^-)$, there exists one and only one Euclidean connection on $o(N^-)$. If ∇^1, ∇^2 are two Euclidean on $o(N^-)$, set $w = \nabla^1 - \nabla^2$, then $w \in C^\infty\big(B, T^*B \otimes \mathrm{End}(o(N^-))\big)$. Since ∇^1, ∇^2 preserve the metric, w is antisymmetric. Thus w equals to 0. Therefore we have $\nabla^{o(N^-)} = d$.

For any $\mu > 0$, let E^μ (resp. $\mathbf{E}^\mu$, resp. F^μ) be the set of sections of $\Lambda(T^*M)$ on M (resp. of $\pi^*\Lambda(T^*M)|_B$ on the total space N, resp. of $\Lambda(T^*B) \otimes o(N^-)$ on B) which lies in the μ-th Sobolev spaces. Let $|\ |_{\mathrm{E}^\mu}$ (resp. $\big|\ \big|_{\mathbf{E}^\mu}$, resp. $\big|\ \big|_{\mathrm{F}^\mu}$) be the Sobolev norm on E^μ (resp. $\mathbf{E}^\mu$, resp. F^μ). We will always assume that the norm $|\cdot|_{\mathrm{E}^0}$ (resp. $\big|\cdot\big|_{\mathbf{E}^0}$) is the norm associated with the Euclidean product (2.3.42) (resp (2.3.46)). The norm $|\cdot|_{\mathrm{F}^0}$ defined on the section of $\Lambda(T^*B)\otimes o(N^-)$ is associated with a Euclidean product similar to (2.3.42).

Take $\varepsilon \in (0, \frac{\varepsilon_0}{2}]$. Let φ be a smooth function on $\mathbb{R}$ with values in [0, 1] such that

$$\varphi(a) = \begin{cases} 1 & \text{if } a \leqslant \frac{1}{2}, \\ 0 & \text{if } a \geqslant 1. \end{cases} \tag{2.3.85}$$

If $y \in B, Z \in N_y$, set

$$\rho(Z) = \varphi(\frac{|Z|}{\varepsilon}). \tag{2.3.86}$$

For $T > 0, y \in B$, set

$$\alpha_T(y) = \int_{N_y} \exp(-T|Z|^2)\rho^2(Z)dv_{N_y}(\mathrm{Z}). \tag{2.3.87}$$

Clearly, for $1 \leqslant i \leqslant r, \alpha_T(y)$ takes a constant value on B_i. We now write α_T instead of $\alpha_T(y)$. Since for $|Z| \leqslant \varepsilon/2,\ \rho(Z) = 1$, there exist $c > 0, C > 0$ such that for $T \geqslant 1$,

$$\frac{c}{T^{l/2}} \leqslant \alpha_T \leqslant \frac{C}{T^{l/2}}. \tag{2.3.88}$$

Here $l = m - n$ denotes the rank of N.

Definition 2.13. For $\mu \geqslant 0, T > 0$, define $J_T : \mathrm{F}^\mu \to \mathbf{E}^\mu$ by: for $s \in \mathrm{F}^\mu$,

$$J_T s(y, Z) = \frac{1}{\sqrt{\alpha_T}}\rho(Z)\exp(-\frac{T|Z|^2}{2})s(y) \wedge \theta_y \in \mathbf{E}^\mu, \tag{2.3.89}$$

where the smooth section θ of $\Lambda^{n^-}(N^-) \otimes o(N^-)$ is given by

$$\theta_y = u^1 \wedge \cdots \wedge u^{n^-} \otimes \Phi(u^1 \wedge \cdots \wedge u^{n^-})$$

for any orthonormal basis $\{u^j\}_{j=1}^{n^-}$ of N_y^-.

It is easy to see that J_T is an isometry from F^0 onto its images. In fact for $s \in \mathrm{F}^0$,

$$\begin{aligned} \|J_T s\|^2_{\mathbf{E}^0} &= \int_B \int_{N_y} \langle J_T s, J_T s\rangle(y, Z)dv_N(y, Z) \\ &= \int_B \frac{1}{\alpha_T} \int_{N_y} \rho^2(Z)\exp(-T|Z|^2)\langle\theta_y, \theta_y\rangle\langle s(y), s(y)\rangle dv_{N_y}(Z)dv_Y(y) \\ &= \int_B \langle s(y), s(y)\rangle dv_Y(y)\frac{1}{\alpha_T}\int_{N_y} \rho^2(Z)\exp(-T|Z|^2)dv_{N_y}(Z) \\ &= \int_B \langle s(y), s(y)\rangle dv_Y(y) \\ &= \|s\|^2_{\mathrm{F}^0}. \end{aligned} \tag{2.3.90}$$

For $\mu \geqslant 0, T > 0$, let $\mathbf{E}_T^\mu$ be the image of F^μ in $\mathbf{E}^\mu$ by J_T. Let $\mathbf{E}_T^{0,\perp}$ be the orthogonal space to $\mathbf{E}_T^0$ in $\mathbf{E}^0$, let $p_T, p_T^\perp$ be the orthogonal projection operators from $\mathbf{E}^0$ on $\mathbf{E}_T^0, \mathbf{E}_T^{0,\perp}$ respectively.

Recall that on $\mathcal{B}_{\varepsilon_0} \simeq \mathcal{U}_{\varepsilon_0}$, $\Lambda(T^*M)$ is identified with $\pi^*(\Lambda T^*M)|_B$. Therefore if $s \in \mathrm{F}^\mu$, we can also consider $J_T s$ as an element of E^μ. Let E_T^μ be the image of F^μ in E^μ by J_T. Particularly $\mathbf{E}_T^0$ may be identified with E_T^0. Moreover, by (2.3.45) the identification is isometric. Let $\mathrm{E}_T^{0,\perp}$ be the orthogonal space to E_T^0 in E^0. Then E^0 splits orthogonally into

$$\mathrm{E}^0 = \mathrm{E}_T^0 \oplus \mathrm{E}_T^{0,\perp}. \tag{2.3.91}$$

Let $\overline{p}_T, \overline{p}_T^\perp$ be the orthogonal projection operators from E^0 on $\mathrm{E}_T^0, \mathrm{E}_T^{0,\perp}$ respectively. We denote by $\mathrm{Supp}(s)$ the support of any section s. Since E_T^0 may be identified isometrically with $\mathbf{E}_T^0$, we find that

$$\overline{p}_T s = p_T s, \text{ for any } s \in \mathrm{E}^0 \text{ and } \mathrm{Supp}(s) \subset \mathcal{B}_{\varepsilon_0}. \tag{2.3.92}$$

In particular,

$$\overline{p}_T J_T s = p_T J_T s, \text{ for any } s \in \mathrm{F}^0. \tag{2.3.93}$$

According to the decomposition (2.3.91) we set:

$$\begin{aligned} D_{T,1} &= \overline{p}_T D_T \overline{p}_T, \quad D_{T,2} = \overline{p}_T D_T \overline{p}_T^\perp, \\ D_{T,3} &= \overline{p}_T^\perp D_T \overline{p}_T, \quad D_{T,4} = \overline{p}_T^\perp D_T \overline{p}_T^\perp. \end{aligned} \tag{2.3.94}$$

Then

$$D_T = D_{T,1} + D_{T,2} + D_{T,3} + D_{T,4}. \tag{2.3.95}$$

We will now establish various estimates on the $D_{T,j}$'s as $T \to +\infty$.

We defined a twisted de-Rham operator

$$D^B = \sum_{j=1}^n c(f_j)\nabla_{f_j}^B : C^\infty(B, \Lambda(T^*B) \otimes o(N^-)) \longrightarrow C^\infty(B, (\Lambda T^*B) \otimes o(N^-)),$$

where $\nabla^B = \nabla^{TB} \otimes 1 + 1 \otimes \nabla^{o(N^-)}$. Similar to [6, Theorem9.8] and [18, Lemma 3.1], we have the following result.

Proposition 2.14. *As $T \to +\infty$, the following formula holds*

$$J_T^{-1} D_{T,1} J_T = D^B + O(\frac{1}{\sqrt{T}}), \tag{2.3.96}$$

where $O(\frac{1}{\sqrt{T}})$ is a first order differential operator with smooth coefficients dominated by $C/\sqrt{T}$.

Proof. We can proceed as in [6, Theorem 9.8], [18, Lemma 3.1] and the proof is easier here due to the fact that the local formula (2.3.81) of gradient of f is simple.

By (2.3.57),

$$D_{T,1} = \overline{p}_T D_T \overline{p}_T = \overline{p}_T (D^H + D_T^N + R_T)\overline{p}_T, \tag{2.3.97}$$

where

$$R_T = O(|Z|^2\partial^H + |Z|^2\partial^N + |Z| + T|Z|^4). \tag{2.3.98}$$

From (2.3.92), (2.3.93) and (2.3.97), we find that

$$J_T^{-1} D_{T,1} J_T = J_T^{-1} p_T (D^H + D_T^N + R_T) p_T J_T. \tag{2.3.99}$$

We may write out the projection p_T explicitly.

From (2.3.89), one verifies directly that for $s \in \mathbf{E}^0$,

$$p_T s(y,Z) = \frac{1}{\alpha_T(y)}\rho(Z)\exp(-\frac{T|Z|^2}{2})\int_{N_y}\langle s(y,Z'),\theta_y\rangle\rho(Z')\exp(-\frac{T|Z'|^2}{2})dv_{N_y}(Z')\wedge\theta_y. \tag{2.3.100}$$

For $s \in \mathrm{E}^0$, $\overline{p}_T s$ is well defined and has the same expression as the above formula for p_T.

From (2.3.15), (2.3.16) and [2, Proposition 1.20], we claim that

$$(\pi^*\nabla^N)_{f_j^H} Z = 0, \quad \nabla^N \theta_y = 0. \tag{2.3.101}$$

In fact from (2.3.25),

$$\begin{aligned}(\pi^*\nabla^N)_{f_j^H} Z =& \sum_{\alpha=1}^{l}\Big[\big(f_j^H Z_\alpha\big)e_\alpha + Z_\alpha(\pi^*\nabla^N)_{f_j^H}e_\alpha\Big] \\ =& -\sum_{\alpha,\beta=1}^{l} w_{y,\beta\alpha}(f_j)Z_\alpha e_\beta + \sum_{\alpha=1}^{l} Z_\alpha(\pi^*\nabla^N)_{f_j}e_\alpha \\ & -\sum_{\alpha,\beta,\gamma=1}^{l} w_{y,\beta\gamma}(f_j)Z_\alpha Z_\gamma(\pi^*\nabla^N)_{e_\beta}e_\alpha\Big] \\ =& \sum_{\alpha,\beta=1}^{l}\Big[-w_{y,\beta\alpha}(f_j)Z_\alpha e_\beta + w_{y,\beta\alpha}(f_j)Z_\alpha e_\beta\Big] \\ =& 0.\end{aligned} \tag{2.3.102}$$

From (2.3.17) and (2.3.102), we find that

$$(\pi^*\nabla^N)_{f_j^H}|Z|^2 = 2\Big\langle(\pi^*\nabla^N)_{f_j^H}Z, Z\Big\rangle = 0, \quad \nabla_{f_j}^{\Lambda N^*\otimes o(N^-)}\theta = 0. \tag{2.3.103}$$

Since $o(N^-)\otimes o(N^-)$ is canonically a trivial real line bundle with trivial metric, given $s\in \mathrm{F}^1$,

$$\begin{aligned}
D^H J_T s(y,Z) =& \sum_{j=1}^n c(f_j)(\pi^*\nabla^{TM|_B})_{f_j^H}\Big[\frac{1}{\sqrt{\alpha_T}}\rho(Z)\exp(-\frac{T|Z|^2}{2})s(y)\wedge\theta_y\Big] \\
=&\frac{1}{\sqrt{\alpha_T}}\sum_{j=1}^n c(f_j)\Big[(\pi^*\nabla^{TM|_B})_{f_j^H}\Big(\rho(Z)\exp(-\frac{T|Z|^2}{2})\Big)s(y)\wedge\theta_y \\
&+\rho(Z)\exp(-\frac{T|Z|^2}{2})s(y)\wedge\pi^*\big(\nabla^{\Lambda N^*\otimes o(N^-)}_{f_j}\theta_y\big) \\
&+\rho(Z)\exp(-\frac{T|Z|^2}{2})\pi^*\big(\nabla^B_{f_j}s(y)\big)\wedge\theta_y\Big] \\
=&\frac{1}{\sqrt{\alpha_T}}\sum_{j=1}^n c(f_j)\rho(Z)\exp(-\frac{T|Z|^2}{2})\pi^*\big(\nabla^B_{f_j}s(y)\big)\wedge\theta_y \\
=&J_T D^B s(y). \qquad (2.3.104)
\end{aligned}$$

By Theorem 2.12, we get for $s\in\mathrm{F}^0$,

$$\begin{aligned}
D_T^N J_T s =&\frac{(-1)^{|s|}}{\sqrt{\alpha_T}}s(y)\wedge D_T^N\Big[\rho(Z)\exp(-\frac{T|Z|^2}{2})\theta_y\Big] \\
=&\frac{(-1)^{|s|}}{\sqrt{\alpha_T}}\exp(-\frac{T|Z|^2}{2})s(y)\wedge c\big(\nabla\rho(Z)\big)\theta_y, \qquad (2.3.105)
\end{aligned}$$

where $\nabla\rho(Z)$ is calculated in the fiber direction, i.e.,

$$\nabla\rho(Z)=\sum_{\alpha=1}^l (e_\alpha\rho)(Z)e_\alpha. \qquad (2.3.106)$$

From (2.3.100), (2.3.105) and (2.3.106), we get that

$$p_T D_T^N p_T J_T s=0. \qquad (2.3.107)$$

For the term containing R_T, we establish that for $T\geqslant 1,\gamma\in\mathbb{R},\ s\in\mathbf{E}^0$,

$$\big\|p_T|Z|^\gamma s\big\|_{\mathbf{E}^0}\leqslant\frac{C}{T^{\frac{\gamma}{2}}}\big\|s\big\|_{\mathbf{E}^0}, \qquad (2.3.108)$$

and that for $s\in\mathrm{F}^1$,

$$\big\|J_T s\big\|_{\mathbf{E}^1}\leqslant C\Big(\big\|s\big\|_{\mathrm{F}^1}+\sqrt{T}\big\|s\big\|_{\mathrm{F}^0}\Big). \qquad (2.3.109)$$

From (2.3.98) and (2.3.108), we have as $T\to\infty$,

$$J_T^{-1}p_T R_T p_T J_T=O(\frac{1}{\sqrt{T}}). \qquad (2.3.110)$$

Now we prove (2.3.108) and (2.3.109). From (2.3.100),

$$\begin{aligned}\big(p_T|Z|^\gamma s\big)(y,Z) =& \frac{1}{\alpha_T(y)}\rho(Z)\exp(-\frac{T|Z|^2}{2})\\ & \int_{N_y}|Z'|^\gamma\Big\langle s(y,Z'),\theta_y\Big\rangle\rho(Z')\exp(-\frac{T|Z'|^2}{2})dv_{N_y}(Z')\wedge\theta_y.\end{aligned}$$

Thus

$$\begin{aligned}\Big|\big(p_T|Z|^\gamma s\big)(y,Z)\Big|^2 \leqslant& \frac{1}{\alpha_T(y)^2}\rho^2(Z)\exp(-T|Z|^2)\\ & \Big|\int_{N_y}|Z'|^\gamma s(y,Z')\rho(Z')\exp(-\frac{T|Z'|^2}{2})dv_{N_y}(Z')\Big|^2\\ \leqslant& \frac{1}{\alpha_T(y)^2}\rho^2(Z)\exp(-T|Z|^2)\\ & \int_{N_y}|Z'|^{2\gamma}\rho^2(Z')\exp(-T|Z'|^2)dv_{N_y}(Z')\int_{N_y}|s(y,Z)|^2dv_{N_y}(Z).\end{aligned} \tag{2.3.111}$$

After changing variables $W=\sqrt{T}Z'$, we obtain that

$$\begin{aligned}\Big|\big(p_T|Z|^\gamma s\big)(y,Z)\Big|^2 \leqslant& \frac{C}{\alpha_T(y)}\rho^2(Z)\exp(-T|Z|^2)\\ & \frac{1}{T^\gamma}\int_{N_y}|W|^{2\gamma}\exp(-|W|^2)dv_{N_y}(W)\int_{N_y}|s(y,Z')|^2dv_{N_y}(Z')\\ \leqslant& \frac{C}{T^\gamma}\frac{1}{\alpha_T(y)}\rho^2(Z)\exp(-T|Z|^2)\int_{N_y}|s(y,Z')|^2dv_{N_y}(Z').\end{aligned} \tag{2.3.112}$$

Then

$$\begin{aligned}\big\|p_T|Z|^\gamma s\big\|^2_{\mathbf{E}^0} =& \int_B\int_{N_y}|p_T|Z|^\gamma s(y,Z)|^2dv_{N_y}(Z)dv_B(y)\\ \leqslant& \frac{C}{T^\gamma}\int_B\frac{1}{\alpha_T(y)}\Big[\int_{N_y}\rho^2(Z)\exp(-T|Z|^2)dv_{N_y}(Z)\int_{N_y}|s(y,Z')|^2dv_{N_y}(Z')\Big]dv_B(y)\\ =& \frac{C}{T^\gamma}\int_B\int_{N_y}|s(y,Z')|^2dv_{N_y}(Z')dv_B(y)\\ =& \frac{C}{T^\gamma}\big\|s\big\|^2_{\mathbf{E}^0}.\end{aligned} \tag{2.3.113}$$

Hence, (2.3.108) holds. The proof of (2.3.109) is similar. From (2.3.104), (2.3.107) and (2.3.110), we finish the proof of Proposition 2.14. □

Remark 2.15. For $s\in\mathbf{E}^1$ with $\mathrm{Supp}(s)\subset\mathcal{B}_{\frac{3}{4}\varepsilon_0}$, we deduce from (2.3.98) and (2.3.108) that for $T\geqslant 1$,

$$\big\|p_TR_Ts\big\|_{\mathbf{E}^0}\leqslant\frac{C}{\sqrt{T}}\big\|s\big\|_{\mathbf{E}^1}. \tag{2.3.114}$$

Set

$$\mathrm{E}_T^{\mu,\perp} = \mathrm{E}^\mu \cap \mathrm{E}_T^{0,\perp}.$$

Lemma 2.16. *There exists $T_0 > 0, C_1 > 0, C_2 > 0$ such that for any $T \geqslant T_0, s \in \mathrm{E}_T^{1,\perp}, s_1 \in \mathrm{E}_T^1$, we have*

$$\begin{aligned} \left\|D_{T,2}s\right\|_{\mathrm{E}^0} \leqslant & \frac{C_1}{\sqrt{\mathrm{T}}}\left\|s\right\|_{\mathrm{E}^1}, \\ \left\|D_{T,3}s_1\right\|_{\mathrm{E}^0} \leqslant & \frac{C_1}{\sqrt{T}}\left\|s_1\right\|_{\mathrm{E}^1}, \\ \left\|D_{T,4}s\right\|_{\mathrm{E}^0} \geqslant & C_2\big(\|s\|_{\mathrm{E}^1} + \sqrt{T}\left\|s\right\|_{\mathrm{E}^0}\big). \end{aligned} \tag{2.3.115}$$

Proof. Let γ be a smooth function on $\mathbb{R}$ with values in [0, 1] such that

$$\gamma(a) = \begin{cases} 1 & \text{if } a \leqslant \frac{1}{2}, \\ 0 & \text{if } a \geqslant \frac{3}{4}. \end{cases}$$

Set

$$\psi(Z) = \gamma\big(\frac{|Z|}{\varepsilon_0}\big). \tag{2.3.116}$$

We will consider ψ as a function defined on M, which vanishes outside $\mathcal{U}_{\frac{3\varepsilon_0}{4}}$.

Take $s \in \mathrm{E}_T^{1,\perp}$. Set

$$\overline{s} = \psi s.$$

Since $\varepsilon \leqslant \frac{\varepsilon_0}{2}$, ψ equals to 1 on the support of ρ. Since $\overline{p}_T s = 0$, we get from (2.3.100) $\overline{p}_T \overline{s} = 0$, i.e., $\overline{s} \in \mathrm{E}_T^{1,\perp}$. Again by (2.3.100) we have $\overline{p}_T D_T s = \overline{p}_T D_T \overline{s}$ as $D_T s = D_T \overline{s}$ on the support of ρ. That is

$$D_{T,2}s = D_{T,2}\overline{s}.$$

For $\overline{s} \in \mathrm{E}_T^{1,\perp}$,

$$D_{T,2}\overline{s}(y,Z) = p_T(D^H + D_T^N + R_T)\overline{s}(y,Z). \tag{2.3.117}$$

For the term with D_T^N, from (2.3.53), (2.3.100) and Theorem 2.12 we find that

$$\begin{aligned}\big(p_T D_T^N \overline{s}\big)(y,Z) =& \frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ & \int_{N_y}\Big\langle D_T^N\overline{s}(y,Z'),\rho(Z')\exp\big(-\frac{T|Z'|^2}{2}\big)\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y\\ =& \frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ & \int_{N_y}\Big\langle \overline{s}(y,Z'),D_T^N\big[\rho(Z')\exp(-\frac{T|Z'|^2}{2})\theta_y\big]\Big\rangle dv_{N_y}(Z')\wedge\theta_y\\ =& \frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ & \int_{N_y}\Big\langle \overline{s}(y,Z'),c\big(\nabla\rho(Z')\big)\exp\big(-\frac{T|Z'|^2}{2}\big)\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y.\end{aligned} \tag{2.3.118}$$

From (2.3.118), we get

$$\begin{aligned}\Big|\big(p_T D_T^N \overline{s}\big)(y,Z)\Big|^2 \leqslant& \frac{1}{\alpha_T^2(y)}\rho^2(Z)\exp\big(-T|Z|^2\big)\cdot\int_{N_y}\big|\overline{s}(y,Z')\big|^2 dv_{N_y}(Z')\\ & \int_{N_y}\Big|c\big(\nabla\rho(Z')\big)\exp\big(-\frac{T|Z'|}{2}\big)\theta_y\Big|^2 dv_{N_y}(Z').\end{aligned} \tag{2.3.119}$$

As $\nabla\rho(Z')=0$ for $|Z'|\leqslant\frac{\varepsilon}{2}$,we get

$$\big\|p_T D_T^N\overline{s}\big\|_{\mathbf{E}^0}\leqslant Ce^{-C'T}\big\|\overline{s}\big\|_{\mathbf{E}^0}\leqslant\frac{C}{\sqrt{T}}\big\|\overline{s}\big\|_{\mathbf{E}^0}\leqslant\frac{C}{\sqrt{T}}\big\|s\big\|_{\mathrm{E}^0}. \tag{2.3.120}$$

Next we claim for any $s'\in\mathbf{E}^1$,

$$p_T D^H s'(y,Z)=D^H p_T s'(y,Z). \tag{2.3.121}$$

In fact from (2.3.100) and (2.3.101), we find that

$$\begin{aligned}D^H p_T s'(y,Z) =& D^H\Big[\frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ & \int_{N_y}\Big\langle s'(y,Z'),\theta_y\Big\rangle\rho(Z')\exp\big(-\frac{T|Z'|^2}{2}\big)dv_{N_y}(Z')\wedge\theta_y\Big]\\ =& \frac{1}{\alpha_T(y)}\rho(Z)\exp(-\frac{T|Z|^2}{2})\sum_{j=1}^n c(f_j)\cdot\\ & \big(\pi^*\nabla^{\Lambda(T^*M)|_B}\big)_{f_j}\Big[\int_{N_y}\Big\langle s'(y,Z'),\rho(Z')\exp\big(-\frac{T|Z'|^2}{2}\big)\theta_y\Big\rangle dv_{N_y}(Z')\Big]\wedge\theta_y.\end{aligned} \tag{2.3.122}$$

On the other hand, from (2.3.100) and (2.3.101),

$$\begin{aligned}\big(p_T D^H s'\big)(y,Z) =&\frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ &\int_{N_y}\Big\langle (D^H s')(y,Z'),\rho(Z')\exp(-\frac{T|Z'|^2}{2})\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y\\ =&\frac{1}{\alpha_T(y)}\rho(Z)\exp(-\frac{T|Z|^2}{2})\\ &\int_{N_y} D^H\Big\langle s'(y,Z'),\rho(Z')\exp(-\frac{T|Z'|^2}{2})\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y. \end{aligned}\tag{2.3.123}$$

By integration by parts and antisymmetry of $w_{y,\alpha\beta}$ as in (2.3.49), we find that for $j=1,\cdots,n$,

$$\begin{aligned}\big(\pi^*\nabla^{TM|_B}_{f_j^H} p_T s'\big)(y,Z) =&\frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ &\big(\pi^*\nabla^{TM|_B}\big)_{f_j}\int_{N_y}\Big\langle s'(y,Z'),\rho(Z')\exp\big(-\frac{T|Z'|^2}{2}\big)\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y\\ =&\frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ &\int_{N_y}\big(\pi^*\nabla^{TM|_B}\big)_{f_j^H}\Big\langle s'(y,Z'),\rho(Z')\exp\big(-\frac{T|Z'|^2}{2}\big)\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y\\ =&\frac{1}{\alpha_T(y)}\rho(Z)\exp\big(-\frac{T|Z|^2}{2}\big)\\ &\int_{N_y}\Big\langle\big(\pi^*\nabla^{TM|_B}\big)_{f_j^H}s'(y,Z'),\rho(Z')\exp\big(-\frac{T|Z'|^2}{2}\big)\theta_y\Big\rangle dv_{N_y}(Z')\wedge\theta_y\\ =&\Big(p_T\big(\pi^*\nabla^{TM|_B}\big)_{f_j^H}s'\Big)(y,Z).\end{aligned}\tag{2.3.124}$$

Thus for the term with D^H in (2.3.117), we have for $\overline{s}\in \mathrm{E}_T^{1,\perp}$,

$$p_T D^H\overline{s}(y,Z)) = D^H p_T\overline{s}(y,Z) = 0. \tag{2.3.125}$$

From (2.3.114), (2.3.120) and (2.3.125), we get the first inequality in (2.3.115).

For $s_1\in\mathrm{E}_T^1$,

$$D_{T,3}s_1(y,Z) = p_T^{\perp}(D^H + D_T^N + R_T)s_1(y,Z). \tag{2.3.126}$$

For every $s'\in\mathbf{E}^1$, $p_T s'\in\mathbf{E}_T^1$ may be viewed as smooth section in E_T^1. It is clear that (2.3.120) holds for $s'\in\mathbf{E}^1$. Note that $D_T^N p_T$ is the formal adjoint operator of $p_T D_T^N$. From (2.3.120) we have

$$\big\|D_T^N p_T s'\big\|_{\mathbf{E}^0}\leqslant\frac{C}{\sqrt{T}}\big\|s'\big\|_{\mathbf{E}^0}. \tag{2.3.127}$$

Then for $s_1 \in \mathrm{E}_T^1$, we have

$$\left\|p_T^{\perp} D_T^N s_1\right\|_{\mathrm{E}^0} \leqslant \left\|D_T^N p_T s_1\right\|_{\mathrm{E}^0} \leqslant \frac{C}{\sqrt{T}}\left\|s_1\right\|_{\mathrm{E}^0}. \tag{2.3.128}$$

For the term with D^H in (2.3.126), from (2.3.121) we find that for $s_1 \in \mathrm{E}_T^1$,

$$p_T^{\perp} D^H s_1 = p_T^{\perp} D^H p_T s_1 = p_T^{\perp} p_T D^H s_1 = 0. \tag{2.3.129}$$

From (2.3.114), (2.3.128) and (2.3.129), we obtain the second inequality in (2.3.115).

To prove the last inequality in (2.3.115), we need to prove the following Proposition about D_T, which we will prove later.

Note that the vector spaces $\mathrm{E}_T^0, \mathrm{E}_T^{0,\perp}$ implicitly depend on $\varepsilon \in (0, \frac{\varepsilon_0}{2}]$.

Proposition 2.17. *There exist $\varepsilon \in (0, \frac{\varepsilon_0}{4}], C > 0, b > 0$ such that for any $T \geqslant 1$, any $s \in \mathrm{E}_T^{1,\perp}$, then*

$$\left\|D_T s\right\|_{\mathrm{E}^0}^2 \geqslant C\Big(\left\|s\right\|_{\mathrm{E}^1}^2 + (T-b)\left\|s\right\|_{\mathrm{E}^0}^2\Big). \tag{2.3.130}$$

Now we continue to prove the last inequality for $D_{T,4}$ in (2.3.115).

For any $s \in \mathrm{E}_T^{1,\perp}$,

$$D_{T,4} s = D_T s - D_{T,2} s.$$

Then

$$\begin{aligned}\left\|D_{T,4} s\right\|_{\mathrm{E}^0}^2 =& \left\|D_T s\right\|_{\mathrm{E}^0}^2 - \left\|D_{T,2} s\right\|_{\mathrm{E}^0}^2 \\ \geqslant& C\Big(\left\|s\right\|_{\mathrm{E}^1}^2 + (T-b)\left\|s\right\|_{\mathrm{E}^0}^2\Big) - \frac{C_1^2}{T}\left\|s\right\|_{\mathrm{E}^0}^2.\end{aligned}$$

So we get last estimate in (2.3.115) for T large enough and the proof of Lemma 2.16 is complete. □

Proof of Proposition 2.17. To prove Proposition 2.17, we need the following two Lemmas. We postpone their proofs later.

Lemma 2.18. *There exist $\varepsilon \in (0, \frac{\varepsilon_0}{4}], C > 0,\ b > 0$ such that for any $T \geqslant 1,\ s \in \mathrm{E}_T^{1,\perp}$ whose support is included in $\mathcal{U}_{2\varepsilon}$, then*

$$\left\|D_T s\right\|_{\mathrm{E}^0}^2 \geqslant C\Big(\left\|s\right\|_{\mathrm{E}^1}^2 + (T-b)\left\|s\right\|_{\mathrm{E}^0}^2\Big). \tag{2.3.131}$$

Throughout the paper, ε may be viewed as a constant once Lemma 2.18 is proved.

Lemma 2.19. *There exist $C > 0,\ b > 0,$ such that for any $T \geqslant 1$, any $s \in \mathrm{E}^1$ which vanishes on $\mathcal{U}_\varepsilon$, then*

$$\left\|D_T s\right\|_{\mathrm{E}^0}^2 \geqslant C\Big(\left\|s\right\|_{\mathrm{E}^1}^2 + (T-b)\left\|s\right\|_{\mathrm{E}^0}^2\Big). \tag{2.3.132}$$

Now using the above two Lemmas, we prove Proposition 2.17 as follows.
Set

$$\begin{aligned}\rho_1(Z) =& \frac{\varphi}{\sqrt{\varphi^2+(1-\varphi)^2}}\left(\frac{|Z|}{2\varepsilon}\right),\\ \rho_2(Z) =& \frac{1-\varphi}{\sqrt{\varphi^2+(1-\varphi)^2}}\left(\frac{|Z|}{2\varepsilon}\right).\end{aligned} \tag{2.3.133}$$

Then $\rho_1,\ \rho_2$ are smooth functions on the total space N such that

$$\rho_1{}^2+\rho_2{}^2=1. \tag{2.3.134}$$

For $j=1,2$, set

$$s_j=\rho_j s. \tag{2.3.135}$$

Then $s_1\in \mathrm{E}_T^{1,\perp}$ and $\mathrm{Supp}(s_1)\subset \mathcal{U}_{2\varepsilon}$ and $s_2\in \mathrm{E}^1$ vanishing on $\mathcal{U}_\varepsilon$.
By (2.3.100) and ρ_1 equals to 1 on support of ρ, we have for $s\in \mathrm{E}_T^{1,\perp}$,

$$p_T s_1=0,\ \text{ i.e., }\ s_1\in \mathrm{E}_T^{1,\perp}. \tag{2.3.136}$$

Note

$$\Big\langle [\rho_j, D_T^2]s, s_j\Big\rangle = \big\langle \rho_j D_T^2 s, s_j\big\rangle - \big\langle D_T^2 s_j, s_j\big\rangle, \tag{2.3.137}$$

From (2.3.134) and (2.3.137), we find that

$$\begin{aligned}\big\|D_T s\big\|_{\mathrm{E}^0}^2 =& \sum_{j=1}^{2}\big\|D_T s_j\big\|_{\mathrm{E}^0}^2+\sum_{j=1}^{2}\Big\langle [\rho_j, D_T^2]s, s_j\Big\rangle\\ =& \sum_{j=1}^{2}\big\|D_T s_j\big\|_{\mathrm{E}^0}^2+\sum_{j=1}^{2}\Big\langle \big[\rho_j,(D^M)^2\big]s, s_j\Big\rangle.\end{aligned} \tag{2.3.138}$$

Since $\big[\rho_j,(D^M)^2\big]$ is a differential operator of order one, for any $\eta>0$ there exists $C_\eta>0$ such that

$$\sum_{j=1}^{2}\Big|\Big\langle \big[\rho_j,(D^M)^2\big]s, s_j\Big\rangle\Big| \leqslant \eta\big\|s\big\|_{\mathrm{E}^1}^2+C_\eta\big\|s\big\|_{\mathrm{E}^0}^2. \tag{2.3.139}$$

From (2.3.138) and (2.3.139), we get

$$\big\|D_T s\big\|_{\mathrm{E}^0}^2 \geqslant \sum_{j=1}^{2}\big\|D_T s_j\big\|_{\mathrm{E}^0}^2-\eta\big\|s\big\|_{\mathrm{E}^1}^2-C_\eta\big\|s\big\|_{\mathrm{E}^0}^2. \tag{2.3.140}$$

Now we use Lemma 2.18 and Lemma 2.19 to find that

$$\begin{aligned}\left\|D_T s\right\|_{\mathrm{E}^0}^2 \geqslant & C\sum_{j=1}^{2}\left\|s_j\right\|_{\mathrm{E}^1}^2 + C(T-b)\sum_{j=1}^{2}\left\|s_j\right\|_{\mathrm{E}^0}^2 - \eta\left\|s\right\|_{\mathrm{E}^1}^2 - C_\eta\|s\|_{\mathrm{E}^0}^2 \\ = & C\sum_{j=1}^{2}\left\|s_j\right\|_{\mathrm{E}^1}^2 - \eta\left\|s\right\|_{\mathrm{E}^1}^2 + (CT - Cb - C_\eta)\|s\|_{\mathrm{E}^0}^2. \end{aligned} \tag{2.3.141}$$

By (2.3.134),

$$\sum_{j=1}^{2}\left\|s_j\right\|_{\mathrm{E}^1}^2 \geqslant \frac{1}{2}\left\|s\right\|_{\mathrm{E}^1}^2 - \tilde{C}\left\|s\right\|_{\mathrm{E}^0}^2. \tag{2.3.142}$$

At last we get

$$\left\|D_T s\right\|_{\mathrm{E}^0}^2 \geqslant (\frac{C}{2} - \eta)\left\|s\right\|_{\mathrm{E}^1}^2 + (CT - Cb - C_\eta - \tilde{C})\left\|s\right\|_{\mathrm{E}^0}^2. \tag{2.3.143}$$

By taking $\eta \leqslant \frac{C}{4}$, we get (2.3.130). The proof of Proposition 2.17 is complete. □

Proof of Lemma 2.19. From (2.3.44), it is clear that

$$D_T^2 = (D^M)^2 + T[D^M, \widehat{c}(\nabla f)] + T^2|\nabla f|^2. \tag{2.3.144}$$

As ∇f is invertible outside of $\mathcal{B}_\varepsilon$, there exists $C > 0$ such that if $s \in \mathrm{E}^1$ vanishes on $\mathcal{B}_\varepsilon$, then

$$\left\|(\nabla f)s\right\|_{\mathrm{E}^0}^2 \geqslant C\left\|s\right\|_{\mathrm{E}^0}^2. \tag{2.3.145}$$

From (2.3.61), we find that

$$\left[D^M, \widehat{c}(\nabla f)\right] = \sum_{j=1}^{n} c(f_j)\widehat{c}\big(\nabla_{\tilde{f}_j}^{TM}\cdot\nabla f\big) + \sum_{\alpha=1}^{l} c(e_\alpha)\widehat{c}\big(\nabla_{e_\alpha}^{TM}\cdot\nabla f\big). \tag{2.3.146}$$

Then $\left[D^M, \widehat{c}(\nabla f)\right]$ is an operator of order zero, there exists $C' > 0$ such that

$$\left|\left\langle [D^M, \widehat{c}(\nabla f)]s, s\right\rangle_{\mathrm{E}^0}\right| \leqslant C'\left\|s\right\|_{\mathrm{E}^0}^2. \tag{2.3.147}$$

Since D^M is an elliptic operator of order 1, there exists $C'', C''' > 0$ such that

$$\left\|D^M s\right\|_{\mathrm{E}^0}^2 \geqslant C''\left\|s\right\|_{\mathrm{E}^1}^2 - C'''\left\|s\right\|_{\mathrm{E}^0}^2. \tag{2.3.148}$$

Combining (2.3.144)-(2.3.148), we get

$$\left\|D_T s\right\|_{\mathrm{E}^0}^2 \geqslant C''\left\|s\right\|_{\mathrm{E}^1}^2 + (CT^2 - C'T - C''')\left\|s\right\|_{\mathrm{E}^0}^2, \tag{2.3.149}$$

from which Lemma 2.19 follows. □

Proof of Lemma 2.18. We must be careful to distinguish whether the estimates below are independent of ε, since the vector spaces $\mathrm{E}_T^0, \mathrm{E}_T^{0,\perp}$ do implicitly depend on ε. We will use $C_3, C_4, \cdots$ to denote constants independent of ε and C_ε to denote constants which do depend on ε.

Since $s \in \mathrm{E}_T^{1,\perp}$ and $\mathrm{Supp}(s) \subset \mathcal{U}_{2\varepsilon}$, it may be regarded as an element of $\mathbf{E}^1$. Then

$$D_T s = D_T^N s + D^H s + R_T s. \tag{2.3.150}$$

Then

$$\big\|D_T s\big\|^2_{\mathbf{E}^0} \geqslant \frac{1}{2}\big\|(D^H + D_T^N)s\big\|^2_{\mathbf{E}^0} - \big\|R_T s\big\|^2_{\mathbf{E}^0}. \tag{2.3.151}$$

It is clear that

$$(D^H + D_T^N)^2 = (D^H)^2 + (D_T^N)^2 + \big[D^H, D_T^N\big]. \tag{2.3.152}$$

We now establish the following relation on $\mathbf{E^1}$:

$$\big[D^H, D^N\big]_{(0,Z)} = 0. \tag{2.3.153}$$

We prove (2.3.153) as follows. Let $h \in C^\infty(N), w \in C^\infty\big(B, \Lambda(T^*M)|_B\big)$. We get

$$D^N(h\pi^* w) = \sum_{\alpha=1}^{l} (e_\alpha h) c(e_\alpha)\pi^* w.$$

Then,

$$\begin{aligned} D^H D^N(h\pi^* w) = \sum_{j=1}^{n}\sum_{\alpha=1}^{l} \Big\{ &\big(f_j^H(e_\alpha h)\big)c(f_j)c(e_\alpha)\pi^* w + (e_\alpha h)c(f_j)\cdot \\ &\Big[c(e_\alpha)\pi^*(\nabla^{TB}_{f_j} w) + c\Big((\pi^*\nabla^{TM|_B})_{f_j^H} e_\alpha\Big)\pi^* w\Big]\Big\}. \end{aligned}$$

On the other hand,

$$D^H(h\pi^* w) = \sum_{j=1}^{n} c(f_j)\Big[(f_j^H h)\pi^* w + h\pi^*(\nabla^{TB}_{f_j} w)\Big].$$

Then

$$D^N D^H(h\pi^* w) = \sum_{j=1}^{n}\sum_{\alpha=1}^{l} \Big[\big(e_\alpha(f_j^H h)\big)c(e_\alpha)c(f_j)\pi^* w + (e_\alpha h)c(e_\alpha)c(f_j)\pi^*(\nabla^{TB}_{f_j} w)\Big].$$

Thus

$$\begin{aligned} \big[D^H, D^N\big](h\pi^* w) = \sum_{j=1}^{n}\sum_{\alpha=1}^{l} \Big\{ &\Big([e_\alpha, f_j^H]h\Big)c(e_\alpha)c(f_j)\pi^* w \\ &+ (e_\alpha h)c(f_j)c\Big((\pi^*\nabla^{TM|_B})_{f_j^H} e_\alpha\Big)\pi^* w\Big\}. \end{aligned}$$

By (2.3.17),

$$(\pi^*\nabla^{TM|_B})_{f_j^H}e_\alpha = \nabla^{TM|_B}_{f_j}e_\alpha = \nabla^N_{f_j}e_\alpha = w_{y,\beta\alpha}(f_j)e_\beta.$$

Moreover from (2.3.25),

$$[e_\alpha, f_j^H] = [e_\alpha, f_j - w(f_j)Z] = -[e_\alpha, w_{y,\beta\gamma}(f_j)Z_\gamma e_\beta] = -w_{y,\beta\alpha}(f_j)e_\beta.$$

Therefore

$$\big[D^H, D^N\big]_{(0,Z)}(h\pi^*w) = 0.$$

One can also verify that

$$\big[D^H, \hat{c}(v)\big]_{(0,Z)} = 0. \qquad (2.3.154)$$

By (2.3.8) and (2.3.25), we get for $j = 1, \cdots, n$,

$$\begin{aligned}(\pi^*\nabla^{TM|_B})_{f_j^H}v =& \sum_{\beta=1}^{n^-}\sum_{\alpha=1}^{l}\big[w_{y,\beta\alpha}(f_j)Z_\alpha e_\beta - w_{y,\alpha\beta}(f_j)Z_\beta e_\alpha\big] \\ &+ \sum_{\beta=n^-+1}^{l}\sum_{\alpha=1}^{l}\big[w_{y,\alpha\beta}(f_j)Z_\beta e_\alpha - w_{y,\beta\alpha}(f_j)Z_\alpha e_\beta\big].\end{aligned}$$

Since $w_{y,\alpha\beta} = 0$ for $1 \leqslant \alpha \leqslant n^-, n^- + 1 \leqslant \beta \leqslant l$, then

$$(\pi^*\nabla^{TM|_B})_{f_j^H}v = 0.$$

Thus (2.3.154) holds. By (2.3.153) and (2.3.154),

$$\big[D^H, D_T^N\big] = 0.$$

Let $\mathrm{E}_T^{\prime,0}$ be the image of F^0 in $\mathbf{E}^0$ by the linear map

$$s \in \mathrm{F}^0 \longmapsto \exp\big(-\frac{T|Z|^2}{2}\big)s \wedge \theta \in \mathbf{E}^0. \qquad (2.3.155)$$

Then $\mathrm{E}_T^{\prime,0}$ is exactly the kernel of D_T^N by Theorem 2.12. Let p'_T be the orthogonal projection operator from E^0 on $\mathrm{E}_T^{\prime,0}$.

Similar as (2.3.100), for $s \in \mathrm{F}^0$ we have

$$\begin{aligned}p'_T s(y,Z) =& (\frac{T}{\pi})^{\frac{l}{2}}\exp\big(-\frac{T|Z|^2}{2}\big) \\ &\int_{N_y}\Big\langle s(y,Z'), \theta_y\Big\rangle \exp\big(-\frac{T|Z'|^2}{2}\big)dv_{N_y}(Z') \wedge \theta_y.\end{aligned} \qquad (2.3.156)$$

Then

$$D_T^N s = D_T^N (s - p'_T s). \tag{2.3.157}$$

Using Theorem 2.12, for $s \in \mathrm{E}_T^{1,\perp}$ whose support is contained in $\mathcal{U}_{2\varepsilon}$, we have

$$\big\|D_T^N s\big\|_{\mathrm{E}^0}^2 \geqslant 2T\big\|s - p'_T s\big\|_{\mathrm{E}^0}^2 \geqslant 2T\Big(\big\|s\big\|_{\mathrm{E}^0}^2 - \big\|p'_T s\big\|_{\mathrm{E}^0}^2\Big). \tag{2.3.158}$$

From (2.3.84), we find that there exists $C_3 \geqslant 0$ such that for any $C_4 \in (0,1]$,

$$\big\|D_T^N s\big\|_{\mathbf{E}^0}^2 \geqslant \frac{C_4}{2}\big\langle \Delta^N s, s\big\rangle_{\mathbf{E}^0} + \frac{C_4}{2}T^2\big\||Z|\cdot s\big\|_{\mathbf{E}^0}^2 - C_3C_4T\big\|s\big\|_{\mathbf{E}^0}^2. \tag{2.3.159}$$

We now fix $C_4 \in (0,1]$ such that $C_3C_4 \leqslant 1$.

From (2.3.158) and (2.3.159), we have

$$\begin{aligned}\big\|D_T^N s\big\|_{\mathbf{E}^0}^2 \geqslant& \frac{1}{4}C_4\big\langle \Delta^N s, s\big\rangle_{\mathbf{E}^0} + \frac{1}{4}C_4T^2\big\||Z|\cdot s\big\|_{\mathbf{E}^0}^2 \\ &+ \frac{1}{2}T\big\|s\big\|_{\mathbf{E}^0}^2 - T\big\|p'_T s\big\|_{\mathbf{E}^0}^2.\end{aligned} \tag{2.3.160}$$

Using elliptic estimates, there exists $C_5 > 0,\ C_6 > 0$ such that

$$\frac{1}{4}C_4\big\langle \Delta^N s, s\big\rangle_{\mathbf{E}^0} + \big\|D^H s\big\|_{\mathbf{E}^0}^2 \geqslant C_5\big\|s\big\|_{\mathbf{E}^1}^2 - C_6\big\|s\big\|_{\mathbf{E}^0}^2. \tag{2.3.161}$$

As $s \in \mathrm{E}_T^{1,\perp}$ has support in $\mathcal{U}_{2\varepsilon}$, $p_T s = 0$. From (2.3.100) and (2.3.156), we find,

$$\begin{aligned}p'_T s(y,Z) =& \big(\frac{T}{\pi}\big)^{\frac{l}{2}}\exp\big(-\frac{T|Z|^2}{2}\big) \\ &\int_{N_y}\big\langle s(y,Z'),\theta_y\big\rangle\big(1-\rho(Z')\big)\exp\big(-\frac{T|Z'|^2}{2}\big)dv_{N_y}(Z')\wedge\theta_y.\end{aligned}$$

The function $1-\rho(Z)$ vanishes for $|Z| \leqslant \varepsilon/2$, thus

$$\big\|p'_T s\big\|_{\mathbf{E}^0}^2 \leqslant \frac{C_\varepsilon}{\sqrt{T}}\big\|s\big\|_{\mathbf{E}^0}^2. \tag{2.3.162}$$

From (2.3.152)–(2.3.154) and (2.3.160)–(2.3.162), we finally get

$$\begin{aligned}\big\|(D^H + D_T^N)s\big\|_{\mathbf{E}^0}^2 \geqslant& C_5\big\|s\big\|_{\mathbf{E}^1}^2 + \frac{1}{4}C_4T^2\big\||Z|s\big\|_{\mathbf{E}^0}^2 \\ &+ \big(\frac{1}{2}T - \sqrt{T}C_\varepsilon - C_6\big)\big\|s\big\|_{\mathbf{E}^0}^2.\end{aligned} \tag{2.3.163}$$

From (2.3.98), there exists $C_7 > 0$ such that

$$\big\|R_T s\big\|_{\mathbf{E}^0}^2 \leqslant C_7\big(\varepsilon^2\big\|s\big\|_{\mathbf{E}^1}^2 + T^2\varepsilon^6\big\||Z|\cdot s\big\|_{\mathbf{E}^0}^2\big). \tag{2.3.164}$$

From (2.3.151), (2.3.163) and (2.3.164), we have

$$\begin{aligned}2\big\|D_T s\big\|_{\mathbf{E}^0}^2 \geqslant& (C_5 - 2C_7\varepsilon^2)\big\|s\big\|_{\mathbf{E}^1}^2 + \big(\frac{1}{4}C_4 - 2C_7\varepsilon^6\big)T^2\big\||Z|\cdot s\big\|_{\mathbf{E}^0}^2 \\ &+ \big(\frac{1}{2}T - \sqrt{T}C_\varepsilon - C_6\big)\big\|s\big\|_{\mathbf{E}^0}^2.\end{aligned} \tag{2.3.165}$$

We finally get (2.3.131) from (2.3.165) for ε small. The proof of Lemma 2.18 is complete.

□

2.3.7 Resolvent estimates

In this subsection, we establish estimates on resolvent of some elliptic operators.

We now fix $\varepsilon \in (0, \frac{\varepsilon_0}{4}]$ once and for all as in Lemma 2.18. Also C_2 denote the positive constant which was determined in Lemma 2.16.

Definition 2.20. Let D'_T be the operator

$$D'_T = \begin{pmatrix} D_{T,1} & 0 \\ 0 & D_{T,4} \end{pmatrix}. \tag{2.3.166}$$

Proposition 2.21. *There exist $T_0 \geqslant 1$, $C > 0$ such that*

(1). For any $T \geqslant T_0$, the operator D'_T is self-adjoint with domain E^1, and the operator $D_{T,4}$ is one to one from $\mathrm{E}_T^{1,\perp}$ into $\mathrm{E}_T^{0,\perp}$.

(2). For any $T \geqslant T_0$, $\lambda \in \mathbb{C}$, $|\lambda| \leqslant \frac{C_2}{2}\sqrt{T}$, $s \in \mathrm{E}_T^{0,\perp}$, then

$$\begin{aligned} \big\|(\lambda - D_{T,4})^{-1}s\big\|_{\mathrm{E}_T^{0,\perp}} &\leqslant \frac{C}{\sqrt{T}}\big\|s\big\|_{\mathrm{E}_T^{0,\perp}}, \\ \big\|(\lambda - D_{T,4})^{-1}s\big\|_{\mathrm{E}_T^{1,\perp}} &\leqslant C\big\|s\big\|_{\mathrm{E}_T^{0,\perp}}. \end{aligned} \tag{2.3.167}$$

Proof. Since D^M is an elliptic operator of order 1, by elliptic estimate we have

$$\big\|s\big\|_{\mathrm{E}^1} \leqslant C\Big(\big\|D^M s\big\|_{\mathrm{E}^0} + \big\|s\big\|_{\mathrm{E}^0}\Big). \tag{2.3.168}$$

By (2.3.44),

$$\big\|D^M s\big\|^2 = \big\langle D_T^2 s, s\big\rangle - T\Big\langle \big[D^M, \hat{c}(\nabla f)\big]s, s\Big\rangle - T^2\Big\langle \big(\hat{c}(\nabla f)\big)^2 s, s\Big\rangle. \tag{2.3.169}$$

Since $\big[D^M, \hat{c}(\nabla f)\big]$ is an operator of order zero,

$$\big\|D^M s\big\|_{\mathrm{E}^0} \leqslant \big\|D_T s\big\|_{\mathrm{E}^0} + C\sqrt{T}\big\|s\big\|_{\mathrm{E}^0}. \tag{2.3.170}$$

Using (2.3.168) and (2.3.170),

$$\big\|s\big\|_{\mathrm{E}^1} \leqslant C\Big(\big\|D_T s\big\|_{\mathrm{E}^0} + \sqrt{T}\big\|s\big\|_{\mathrm{E}^0}\Big). \tag{2.3.171}$$

Using Lemma 2.16, we find that if $s \in \mathrm{E}^1$,

$$\big\|(D_T - D'_T)s\big\|_{\mathrm{E}^0} \leqslant \frac{C}{\sqrt{T}}\big\|s\big\|_{\mathrm{E}^1}. \tag{2.3.172}$$

From (2.3.171), (2.3.172), we get for $s \in \mathrm{E}^1$,

$$\big\|(D_T - D'_T)s\big\|_{\mathrm{E}^0} \leqslant C'(\frac{1}{\sqrt{T}}\big\|D_T s\big\|_{\mathrm{E}^0} + \big\|s\big\|_{\mathrm{E}^0}). \tag{2.3.173}$$

For $T \geqslant 1$ large enough, $C'/\sqrt{T}$ is strictly small than 1. Also for any $T \geqslant 1$, D_T is a self-adjoint operator with domain E^1. By the Kato-Rellich theorem, i.e., in [41, Theorem X. 12], we deduce that for $T \geqslant 1$ large enough, the operator D'_T is self-adjoint with domain E^1. In particular for $T \geqslant 1$ large enough, $D_{T,4}$ is self-adjoint with domain $\mathrm{E}_T^{1,\perp}$. From Lemma 2.16, we see that for T large enough, $D_{T,4}$ is one to one from $\mathrm{E}_T^{1,\perp}$ into $\mathrm{E}_T^{0,\perp}$.

The first line in (2.3.167) follows from Lemma 2.16.

By Lemma 2.16, for $\lambda \leqslant \frac{C_2}{2}\sqrt{T}$, $s \in \mathrm{E}_T^{0,\perp}$,

$$\big\|\lambda D_{T,4}^{-1} s\big\|_{\mathrm{E}_T^{0,\perp}} \leqslant \frac{|\lambda|}{C_2\sqrt{T}}\big\|s\big\|_{\mathrm{E}_T^{0,\perp}} \leqslant \frac{1}{2}\big\|s\big\|_{\mathrm{E}_T^{0,\perp}}. \tag{2.3.174}$$

Then

$$\Big\|\frac{D_{T,4}}{\lambda - D_{T,4}} s\Big\|_{\mathrm{E}_T^{0,\perp}} = \big\|(1-\lambda D_{T,4}^{-1})^{-1} s\big\|_{\mathrm{E}_T^{0,\perp}} \leqslant 2\big\|s\big\|_{\mathrm{E}_T^{0,\perp}}. \tag{2.3.175}$$

Again by Lemma 2.16, we have

$$\big\|(\lambda - D_{T,4})^{-1} s\big\|_{\mathrm{E}_T^{1,\perp}} \leqslant \frac{1}{C_2}\Big\|\Big(\frac{D_{T,4}}{\lambda - D_{T,4}}\Big) s\Big\|_{\mathrm{E}_T^{0,\perp}} \leqslant C\big\|s\big\|_{\mathrm{E}_T^{0,\perp}}. \tag{2.3.176}$$

The proof of Proposition 2.21 is complete. □

Definition 2.22. Let H, H$'$ be separable Hilbert spaces. We denote by $\mathcal{L}(\mathrm{H}, \mathrm{H}')$ linear bounded operators from H to H$'$. When H$'$=H, we simply denote by $\mathcal{L}(\mathrm{H})$. For $A \in \mathcal{L}(\mathrm{H}, \mathrm{H}')$, set $|A| = \big(A^*A\big)^{\frac{1}{2}}$. If $1 \leqslant p < +\infty$, set

$$\mathcal{L}_p(\mathrm{H}, \mathrm{H}') = \Big\{A \in \mathcal{L}\big(\mathrm{H}, \mathrm{H}'\big); \mathrm{Tr}\big(|A|^p\big) < +\infty\Big\},$$

If $A \in \mathcal{L}_p\big(\mathrm{H}, \mathrm{H}'\big)$, set

$$\big\|A\big\|_p = \Big[\mathrm{Tr}\big(|A|^p\big)\Big]^{1/p}.$$

Then by [41, Theorem IX] ([42, Theorem 2.3.8]), $\big\|\cdot\big\|_p$ is a norm on $\mathcal{L}_p\big(\mathrm{H}, \mathrm{H}'\big)$. We shall adopt the convention that $\mathcal{L}_\infty\big(\mathrm{H}, \mathrm{H}'\big)$ is $\mathcal{L}\big(\mathrm{H}, \mathrm{H}'\big)$. If $A \in \mathcal{L}\big(\mathrm{H}, \mathrm{H}'\big)$, let $\big\|A\big\|_\infty$ be the usual operator norm of A.

In the sequel, the norms $\big\|\cdot\big\|_p$, $\big\|\cdot\big\|_\infty$ will always be calculated with respect to the Sobolev spaces of order zero like $\mathrm{E}_T^0, \mathrm{E}_T^{0,\perp}, \mathrm{F}^0$.

Now we establish an Lemma as follows.

Lemma 2.23. *If* $p \geqslant 2m+1$, $(D^M + \sqrt{-1})^{-1} \in \mathcal{L}_p\Big(L^2\big(\Omega^{\cdot}(M)\big), L^2\big(\Omega^{\cdot}(M)\big)\Big)$.

Proof. It is clear that $D^M-\sqrt{-1}$ is the adjoint operator of $D^M+\sqrt{-1}$. Set $P=(D^M)^2+1$. P is a generalized Laplacian operator. Recall that $P^{-1}:\mathrm{L}^2\big(\Omega^\cdot(M)\big)\to\mathrm{L}^2\big(\Omega^\cdot(M)\big)$ is continuous. Let $P^{-p/2}(x,y)$ be the Schwartz kernel associated to the operator $P^{-p/2}$. We now prove that $P^{-p/2}(x,y)$ is continuous for $x,y\in M$. Let $k=\frac{p}{2}\geqslant m+\frac{1}{2}$ and $r>0$ such that $2r=\frac{m+1}{2}\leqslant k$. From regularity theorem and Sobolev inequalities, we have

$$\begin{aligned}\big\|P^{-k}P^r s\big\|_{C^0(M)}\leqslant& C\cdot\big\|P^{-k}P^r s\big\|_{\mathrm{E}^{\frac{m+1}{2}}}\\ \leqslant& C\cdot(\big\|P^rP^{-k}P^r s\big\|_{\mathrm{E}^0}+\big\|P^{-k}P^r s\big\|_{\mathrm{E}^0})\\ \leqslant& C\cdot\big\|s\big\|_{\mathrm{E}^0}.\end{aligned}\tag{2.3.177}$$

Here $2r=\frac{m+1}{2}$ is required for the second inequality. From the above inequality, the continuity of $P^{-p/2}(x,y)$ follows.

Then

$$\big\|(D^M+\sqrt{-1})^{-1}\big\|_p^p=\mathrm{Tr}\big(P^{-p/2}\big)=\int_M\mathrm{Tr}\big(P^{-p/2}(x,x)\big)<\infty.\tag{2.3.178}$$

□

Let H be an Hilbert space. Assume $p>2m+1$ and $0<q<p$ satisfies $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 2.24 ([42])**.** *For $k\geqslant 1$, if $A\in\mathcal{L}_{pk}\big(\mathrm{H}\big),B\in\mathcal{L}_{pk}\big(\mathrm{H}\big)$, then $AB\in L_k\big(\mathrm{H}\big)$ and*

$$\big\|AB\big\|_k\leqslant\big\|A\big\|_{kp}\big\|B\big\|_{kq}.\tag{2.3.179}$$

We omit the proofs of the above Lemma. See [42] for more details.

Recall that T_0 is determined in Proposition 2.21.

Proposition 2.25. *If $p\geqslant 2m+1$, there exists $C>0$ such that for $T>T_0$, $\lambda\in\mathbb{C}$, $|\lambda|\leqslant\frac{C_2}{2}\sqrt{T}$, then*

$$\begin{aligned}&\big\|(\lambda-D_{T,4})^{-1}\big\|_\infty\leqslant\frac{C}{\sqrt{T}},\\ &\big\|(\lambda-D_{T,4})^{-1}\big\|_p\ \leqslant C,\\ &\big\|D_{T,2}(\lambda-D_{T,4})^{-1}\big\|_\infty\leqslant\frac{C}{\sqrt{T}}.\end{aligned}\tag{2.3.180}$$

Proof. The first line of (2.3.180) follows from Proposition 2.3.13. Also by Lemma 2.24,

$$\big\|(\lambda-D_{T,4})^{-1}\big\|_p\leqslant\big\|(D^M+\sqrt{-1})^{-1}\big\|_p\big\|(D^M+\sqrt{-1})(\lambda-D_{T,4})^{-1}\big\|_\infty.\tag{2.3.181}$$

From Lemma 2.23, we know that when $p\geqslant 2m+1$, $\big\|(D^M+\sqrt{-1})^{-1}\big\|_p<\infty$.

Also by Proposition 2.21, for $T\geqslant T_0$,

$$\big\|(D^M+\sqrt{-1})(\lambda-D_{T,4})^{-1}\big\|_\infty\leqslant C.\tag{2.3.182}$$

The second line in (2.3.180) follows. Using (2.3.115), (2.3.167), we get the third line in (2.3.180). □

Next we derive the estimates on the resolvent of D^B.

Recall $C_2 > 0$ is the constant determined in Lemma 2.16. Let $C_0 \in (0,1]$ be a constant fixed once and for all such that

$$C_0 < \frac{C_2^2}{9} T_0 \text{ and } \mathrm{Spec}\,(D^B) \cap \Big[-2\sqrt{C_0}, 2\sqrt{C_0} \Big] \subset \{0\}, \tag{2.3.183}$$

where $\mathrm{Spec}(\cdot)$ denotes the spectrum of an operator.

Set

$$\mathrm{U} = \Big\{ \lambda \in \mathbb{C}; |\lambda| < \frac{3}{2}\sqrt{C_0}; \inf_{\mu \in \mathrm{Spec}(D^B)} |\mu - \lambda| \geqslant \frac{\sqrt{C_0}}{4} \Big\}. \tag{2.3.184}$$

Then for any $\lambda \in \mathrm{U}$, $|\lambda| < \frac{C_2}{2}\sqrt{T_0}$.

Proposition 2.26. *There exists a constant $C > 0$ such that for $T \geqslant T_0$, $\lambda \in U$, then*

$$\big\| D_{T,3}(\lambda - J_T D^B J_T^{-1})^{-1} \big\|_\infty \leqslant C. \tag{2.3.185}$$

Proof. Clearly if $\lambda \in \mathrm{U}$,

$$\big\| (\lambda - D^B)^{-1} \big\|_\infty \leqslant C. \tag{2.3.186}$$

Since J_T is an isometry from F^0 into E_T^0, we get

$$\big\| (\lambda - J_T D^B J_T^{-1})^{-1} \big\|_\infty \leqslant C. \tag{2.3.187}$$

By the resolvent equation, we find that

$$(\lambda - D^B)^{-1} = (\sqrt{-1} - D^B)^{-1} + (\sqrt{-1} - \lambda)(\lambda - D^B)^{-1}(\sqrt{-1} - D^B)^{-1}. \tag{2.3.188}$$

Using Sobolev inequalities, if $\sigma \in \mathrm{F}^0$,

$$\big\| (\sqrt{-1} - D^B)^{-1}\sigma \big\|_{\mathrm{F}^1} \leqslant C\Big(\big\|\sigma\big\|_{\mathrm{F}^0} + \big\| (\sqrt{-1} - D^B)^{-1}\sigma \big\|_{\mathrm{F}^0} \Big) \leqslant C\big\|\sigma\big\|_{\mathrm{F}^0}. \tag{2.3.189}$$

From (2.3.188), (2.3.189) and elliptic estimate, we have

$$\begin{aligned} &\big\| (\lambda - D^B)^{-1}\sigma \big\|_{\mathrm{F}^1} \\ \leqslant &\big\| (\sqrt{-1} - D^B)^{-1}\sigma \big\|_{\mathrm{F}^1} + (1 + |\lambda|)\big\| (\sqrt{-1} - D^B)^{-1}(\lambda - D^B)^{-1}\sigma \big\|_{\mathrm{F}^1} \\ \leqslant &C\big\|\sigma\big\|_{\mathrm{F}^0} + C(1 + |\lambda|)\big\| (\lambda - D^B)^{-1}\sigma \big\|_{\mathrm{F}^0} \\ \leqslant &C'(1 + |\lambda|)\big\|\sigma\big\|_{\mathrm{F}^0}. \end{aligned} \tag{2.3.190}$$

Using (2.3.109), (2.3.186) and (2.3.190), we get that if $\lambda \in \mathrm{U}, s \in \mathrm{E}_T^0$,

$$\begin{aligned} \big\| (\lambda - J_T D^B J_T^{-1})^{-1} s \big\|_{\mathrm{E}_T^1} \leqslant &C\Big(\big\| (\lambda - D^B)^{-1} J_T^{-1} s \big\|_{\mathrm{F}^1} + \sqrt{T}\big\| s \big\|_{\mathrm{E}^0} \Big) \\ \leqslant &C\Big[C'\big(1 + |\lambda|\big)\big\| J_T^{-1} s \big\|_{\mathrm{F}^0} + \sqrt{T}\big\| s \big\|_{\mathrm{E}^0} \Big] \\ \leqslant &C''(1 + \sqrt{T})\big\| s \big\|_{\mathrm{E}_T^0}. \end{aligned} \tag{2.3.191}$$

From Lemma 2.16 and (2.3.191), we get (2.3.185). □

By Proposition 2.21, for $T \geqslant T_0, \lambda \in \mathrm{U}$, the operator $\lambda - D_{T,4}$ is an invertible operator from $\mathrm{E}_T^{1,\perp}$ into $\mathrm{E}_T^{0,\perp}$.

Definition 2.27. For $T \geqslant T_0, \lambda \in \mathrm{U}$, let $M_T(\lambda)$ be the linear map from E_T^1 into E_T^0

$$M_T(\lambda) = \lambda - D_{T,1} - D_{T,2}(\lambda - D_{T,4})^{-1} D_{T,3}. \tag{2.3.192}$$

If $s \in \mathrm{E}^0$, set

$$s_1 = \overline{p}_T s;\ \ s_2 = \overline{p}_T^{\perp} s. \tag{2.3.193}$$

Then $s = s_1 + s_2$.

Take then $T \geqslant T_0, \lambda \in \mathrm{U},\ s \in \mathrm{E}^1, s' \in \mathrm{E}^0$. Consider the equation

$$(\lambda - D_T)s = s'. \tag{2.3.194}$$

It is clear that (2.3.194) is equivalent to

$$\begin{aligned} M_T(\lambda)s_1 =& s_1' + D_{T,2}(\lambda - D_{T,4})^{-1} s_2', \\ s_2 =& (\lambda - D_{T,4})^{-1}(s_2' + D_{T,3} s_1). \end{aligned} \tag{2.3.195}$$

From (2.3.195), we deduce that to estimate $(\lambda - D_T)^{-1}$, we need first estimate $M_T^{-1}(\lambda)$.

Theorem 2.28. *There exists $T_1 \geqslant T_0$ such that if $T \geqslant T_1, \lambda \in U$, then $M_T(\lambda)$ is invertible and moreover for any integer $p \geqslant 2m+1$, there exists $C > 0$ such that for $T \geqslant T_1, \lambda \in U$*

$$\begin{aligned} &\left\| M_T^{-1}(\lambda) \right\|_\infty \leqslant C, \\ &\left\| D_{T,3} M_T^{-1}(\lambda) \right\|_\infty \leqslant C, \\ &\left\| M_T^{-1}(\lambda) \right\|_p \leqslant C(1 + |\lambda|), \\ &\left\| J_T^{-1} (M_T^{-1}(\lambda))^p J_T - (\lambda - D^B)^{-p} \right\|_1 \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|)^{p+1}. \end{aligned} \tag{2.3.196}$$

Proof. Set

$$C_T = J_T^{-1} D_{T,1} J_T - D^B. \tag{2.3.197}$$

For $\lambda \in \mathrm{U}$, set

$$\begin{aligned} m_T(\lambda) = 1 - & J_T C_T (\lambda - D^B)^{-1} J_T^{-1} \\ & - D_{T,2}(\lambda - D_{T,4})^{-1} D_{T,3} (\lambda - J_T D^B J_T^{-1})^{-1}. \end{aligned} \tag{2.3.198}$$

Clearly

$$M_T(\lambda) = m_T(\lambda)(\lambda - J_T D^B J_T^{-1}). \tag{2.3.199}$$

Now by Proposition 2.14 and (2.3.190), we find that

$$\left\|C_T(\lambda - D^B)^{-1}\right\|_\infty \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|). \tag{2.3.200}$$

Also by Proposition 2.25 and 2.26, we get

$$\left\|D_{T,2}(\lambda - D_{T,4})^{-1}D_{T,3}(\lambda - J_T D^B J_T^{-1})^{-1}\right\|_\infty \leqslant \frac{C'}{\sqrt{T}}. \tag{2.3.201}$$

From (2.3.198)–(2.3.201), it is clear that if $1/\sqrt{T}$ is small enough, then the operator $m_T(\lambda)$ is invertible, and moreover for $T \geqslant 1$,

$$\left\|m_T^{-1}(\lambda) - 1\right\|_\infty \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|). \tag{2.3.202}$$

In particular,

$$\left\|m_T^{-1}(\lambda)\right\|_\infty \leqslant C'. \tag{2.3.203}$$

By (2.3.199), we get

$$M_T^{-1}(\lambda) = (\lambda - J_T D^B J_T^{-1})^{-1} m_T^{-1}(\lambda). \tag{2.3.204}$$

From (2.3.186), (2.3.203), we obtain the first inequality in (2.3.196). The second inequality in (2.3.196) follows from (2.3.185), (2.3.203) and (2.3.204). From Lemma 2.24 and (2.3.203), we get

$$\left\|M_T^{-1}(\lambda)\right\|_p \leqslant C'\left\|(\lambda - D^B)^{-1}\right\|_p. \tag{2.3.205}$$

From Lemma 2.23, (2.3.188) and Lemma 2.24, we find that if $\lambda \in \mathrm{U}$,

$$\begin{aligned} &\left\|(\lambda - D^B)^{-1}\right\|_p \\ \leqslant &\left\|(\sqrt{-1} - D^B)^{-1}\right\|_p + 2(1+|\lambda|)\left\|(\lambda - D^B)^{-1}\right\|_\infty \cdot \left\|(\sqrt{-1} - D^B)^{-1}\right\|_p \\ \leqslant &C + C(1+|\lambda|). \end{aligned} \tag{2.3.206}$$

The third inequality in (2.3.196) follows from (2.3.205) and (2.3.206). Now we apply Lemma 2.24 to prove the last inequality in (2.3.196) as follows.

Note

$$J_T^{-1} M_T^{-1} J_T = (\lambda - D^B)^{-1}\left[J_T^{-1} m_T^{-1} J_T - 1\right] + (\lambda - D^B)^{-1}. \tag{2.3.207}$$

Set

$$A = J_T^{-1} M_T^{-1} J_T, B = (\lambda - D^B)^{-1}, C = J_T^{-1} m_T^{-1} J_T - 1. \tag{2.3.208}$$

Then $A = BC + B$ and

$$A^p - B^p = \sum_{\substack{A_j \in \{B, BC\}, 1 \leqslant j \leqslant p \\ \sharp\{j | A_j = BC\} \geqslant 1}} A_1 \cdots A_p. \tag{2.3.209}$$

Here $\sharp$ stands for the cardinality of the mentioned finite set. By applying (2.3.179) $p-1$ times, we find that

$$\begin{aligned} \left\|A_1 \cdots A_p\right\|_1 &\leqslant \left\|A_1 \cdots A_{p-1}\right\|_{\frac{p}{p-1}} \left\|A_p\right\|_p \\ &\leqslant \left\|A_1 \cdots A_{p-2}\right\|_{\frac{p}{p-2}} \left\|A_{p-1}\right\|_p \left\|A_p\right\|_p \\ &\cdots \\ &\leqslant \left\|A_1\right\|_p \cdots \left\|A_p\right\|_p. \end{aligned} \tag{2.3.210}$$

We may assume that $A_1 = BC$.
From (2.3.202) and (2.3.206), we get

$$\left\|B\right\|_p \leqslant C(1+|\lambda|), \quad \left\|A_1\right\|_p \leqslant \left\|B\right\|_p \left\|C\right\|_\infty \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|)^2. \tag{2.3.211}$$

If BC appears in A_j more than two times, instead of (2.3.202), we may use the following estimates for other $\left\|C\right\|_\infty$: for $\lambda \in \mathrm{U}$,

$$\left\|C\right\|_\infty \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|) \leqslant C.$$

So

$$\left\|A_1 \cdots A_p\right\|_1 \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|)^{p+1}, \tag{2.3.212}$$

and

$$\left\|A^p - B^p\right\|_1 \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{2.3.213}$$

The proof of last inequality in (2.3.196) is complete. □

If $B \in \mathcal{L}(\mathrm{E}^0)$, for any $T \geqslant 1$, we write B as a matrix with respect to the splitting $E^0 = \mathrm{E}^0_T \oplus \mathrm{E}^{0,\perp}_T$ in the form

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Definition 2.29. If $B \in \mathcal{L}(\mathrm{E}^0), C \in \mathcal{L}(\mathrm{F}^0)$, set

$$d(B, C) = \sum_{j=2}^{4} \left\|B_j\right\|_1 + \left\|J_T^{-1} B_1 J_T - C\right\|_1. \tag{2.3.214}$$

Clearly, if $B \in \mathcal{L}_1(\mathrm{E}^0), C \in \mathcal{L}_1(\mathrm{F}^0)$, then

$$\big|\mathrm{Tr}(B) - \mathrm{Tr}(C)\big| \leqslant d(B, C). \tag{2.3.215}$$

We prove (2.3.215) as follows. If σ_j is an orthonormal basis of F^0, then $J_T\sigma_j$ is an orthonormal basis of E^0_T and $\mathrm{Tr}(B_1) = \mathrm{Tr}(J_T^{-1}B_1J_T)$, so

$$\begin{aligned}\big|\mathrm{Tr}(B) - \mathrm{Tr}(C)\big| \leqslant&\big|\mathrm{Tr}(B_1) - \mathrm{Tr}(C)\big| + \big|\mathrm{Tr}(B_4)\big| \\ =&\big|\mathrm{Tr}(J_T^{-1}B_1J_T) - \mathrm{Tr}(C)\big| + \big|\mathrm{Tr}(B_4)\big| \\ \leqslant&\big\|J_T^{-1}B_1J_T - C\big\|_1 + \big\|B_4\big\|_1. \end{aligned}\tag{2.3.216}$$

Here we have used $\big|\mathrm{Tr}(A)\big| \leqslant \big\|A\big\|_1$, for $A \in \mathcal{L}_1\big(\mathrm{H}\big)$ ([42, Lemma 2.3.3]).

Theorem 2.30. *There exists $T_1 \geqslant T_0$ such that for any $T \geqslant T_1, \lambda \in U$, $\lambda - D_T$ is invertible. For any integer $p \geqslant 2m+2$, there exists $C > 0$ such that if $T \geqslant T_1, \lambda \in U$, then*

$$d\Big((\lambda - D_T)^{-p}, (\lambda - D^B)^{-p}\Big) \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{2.3.217}$$

Proof. Set

$$B_T = (\lambda - D_T)^{-1}. \tag{2.3.218}$$

In view of (2.3.195), we find that

$$\begin{aligned} B_{T,1} &= M_T^{-1}(\lambda), \\ B_{T,2} &= M_T^{-1}(\lambda)D_{T,2}(\lambda - D_{T,4})^{-1}, \\ B_{T,3} &= (\lambda - D_{T,4})^{-1}D_{T,3}M_T^{-1}(\lambda), \\ B_{T,4} &= (\lambda - D_{T,4})^{-1}(1 + D_{T,3}B_{T,2}). \end{aligned}\tag{2.3.219}$$

If $\lambda \in \mathrm{U}$, then $\lambda \leqslant \frac{C_2}{2}\sqrt{T}$. Using Proposition 2.25 and Theorem 2.28, we find that if $p \geqslant 2m+2$, $T \geqslant T_0, \lambda \in \mathrm{U}$, for $j = 2, 3, 4$, then

$$\big\|B_{T,j}\big\|_{p-1} \leqslant C; \quad \big\|B_{T,j}\big\|_\infty \leqslant \frac{C}{\sqrt{T}}. \tag{2.3.220}$$

From (2.3.179), (2.3.220) and Theorem 2.28, we deduce that if $j_1, \cdots, j_p \in \{1, 2, 3, 4\}$, if one of the j's is not equal to 1, suppose there exists $j_i \neq 1$.

$$\begin{aligned}\big\|B_{T,j_1}\cdots B_{T,j_p}\big\|_1 =&\big\|B_{T,j_1}\cdots B_{T,j_{i-1}}(B_{T,j_i}B_{T,j_{i+1}})B_{T,j_{i+2}}\cdots B_{T,j_p}\big\|_1 \\ \leqslant&\big\|B_{T,j_1}\big\|_{p-1}\cdots\big\|B_{T,j_i}B_{T,j_{i+1}}\big\|_{p-1}\cdots\big\|B_{T,j_p}\big\|_{p-1} \\ \leqslant&\big\|B_{T,j_i}\big\|_\infty\cdot\big\|B_{T,j_1}\big\|_{p-1}\cdots\big\|B_{T,j_{i+1}}\big\|_{p-1}\cdots\big\|B_{T,j_p}\big\|_{p-1} \\ \leqslant&\frac{C}{\sqrt{T}}\cdot C(1+|\lambda|)\cdots C(1+|\lambda|)\cdots C(1+|\lambda|) \\ \leqslant&\frac{C}{\sqrt{T}}(1+|\lambda|)^{p-1}. \end{aligned}\tag{2.3.221}$$

Now(2.3.217) immediately follows from the last inequality of (2.3.196) and (2.3.221). □

2.3.8 Proof of Proposition 2.8

Recall the constant $C_0 \in (0,1]$ is fixed such that

$$\text{Spec } (D^B) \cap \Big[-2\sqrt{C_0}, 2\sqrt{C_0} \Big] \subset \{0\}.$$

Let $\mathrm{F}_T^{C_0}$ be the direct sum of eigenspaces of D_T^2 associated to the eigenvalues λ such that $\lambda < C_0$. Let γ be the circle in $\mathbb{C}$ of center 0 and radius $\sqrt{C_0}$. Then $\gamma \subset \mathrm{U}$.

Using Theorem 2.30, we see that for T large enough,

$$\gamma \cap \mathrm{Spec}(D_T) = \emptyset.$$

Set

$$P_T^{C_0} = \frac{1}{2\pi i} \int_\gamma (\lambda - D_T)^{-1} d\lambda. \tag{2.3.222}$$

For T large enough, $P_T^{C_0}$ is exactly the orthogonal projection operator from E^0 on $\mathrm{F}_T^{C_0}$. Let Q be the orthogonal projection operator from F^0 to $K = \mathrm{Ker}\ D^B$. Then we also have the following:

Proposition 2.31. *For T large enough, we have*

$$d(P_T^{C_0}, Q) \leqslant \frac{C}{\sqrt{T}}. \tag{2.3.223}$$

Proof. It is clear that

$$P_T^{C_0} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{p-1} (\lambda - D_T)^{-p} d\lambda, \tag{2.3.224}$$

and

$$Q = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{p-1} (\lambda - D^B)^{-p} d\lambda. \tag{2.3.225}$$

Then

$$d(P_T^{C_0}, Q) \leqslant \frac{1}{2\pi} \int_\gamma |\lambda|^{p-1} d\Big((\lambda - D_T)^{-p}, (\lambda - D^B)^{-p} \Big) d\lambda. \tag{2.3.226}$$

Then (2.3.223) follows form (2.3.217). □

Proof of Proposition 2.8. From (2.3.215) and (2.3.223), we see that for T large enough,

$$\dim \mathrm{F}_T^{C_0} = \dim K. \tag{2.3.227}$$

Let P_j denote the orthogonal projection operator from E^0 onto the L^2-completion space of $\Omega^j(M)$. To prove Proposition 2.8, we need to show that when T is large enough,

$$\dim P_j\ \mathrm{F}_T^{C_0} = q_j. \tag{2.3.228}$$

By (2.3.26),

$$\begin{aligned}\sum_{j=0}^{m}\dim\, P_j\mathrm{F}_T^{C_0} \leqslant \sum_{j=0}^{m}\dim\, \mathrm{F}_T^{C_0} &= \sum_{j=0}^{m}\dim K \\ &= \sum_{i=1}^{r}\sum_{j=0}^{n_i}\dim H^j(B_i, o(N_i^-)) = \sum_{j=0}^{m} q_j. \end{aligned} \tag{2.3.229}$$

Also we find that for any $s_j \in F_j$, $\|s_j\|_{\mathrm{F}^0} = 1$,

$$\begin{aligned}\left\|P_jP_T^{C_0}J_Ts_j - J_Ts_j\right\|_{\mathbf{E}^0} \leqslant& \left\|\left(P_T^{C_0}\right)_1 J_Ts_j - J_Ts_j\right\|_{\mathbf{E}^0} + \left\|\left(P_T^{C_0}\right)_3 J_Ts_j\right\|_{\mathbf{E}^0} \\ \leqslant& \left\|J_T^{-1}(P_T^{C_0})_1J_T - Q\right\|_\infty + \left\|\left(P_T^{C_0}\right)_3\right\|_\infty \\ \leqslant& d(P_T^{C_0}, Q).\end{aligned}$$

Thus from (2.3.223), we have for $s \in K$,

$$\left\|P_jP_T^{C_0}J_Ts - J_Ts\right\|_{\mathbf{E}^0} \leqslant \frac{C}{\sqrt{T}}\left\|s\right\|_{\mathrm{F}^0}. \tag{2.3.230}$$

From (2.3.230), one deduces that

$$\dim\, P_j\ \mathrm{F}_T^{C_0} \geqslant q_j. \tag{2.3.231}$$

From (2.3.229) and (2.3.231), we get (2.3.228). We finish the proof of Proposition 2.8 and hence the proof of Morse-Bott inequalities (2.3.1) and (2.3.2) is complete. □

2.4 Equivariant Morse Inequalities

Now we consider Theorem 2.1 in general case, i.e., G is a finite group. The main goal of this Section is to prove our main Theorem 2.1, i.e., (2.1.12), (2.1.13).

For $p \in B, X \in T_pM, g \in G$, we have

$$\left(g^{-1}\cdot d^2f\right)_p(X,X) = g^{-1}\cdot(d^2f)_{g\cdot p}(X,X) = (d^2f)_{g\cdot p}(g\cdot X, g\cdot X). \tag{2.4.1}$$

On the other hand,

$$\begin{aligned}(g^{-1}\cdot d^2f)_p(X,X) =& g^{-1}\cdot(d^2f)_{g\cdot p}(X,X) \\ =& (g\cdot X)_{gp}\left[(g\cdot\tilde{X})f\right] = X_p\left\{\left[(g\cdot\tilde{X})\cdot f\right]\circ g\right\} \\ =& X_p\left[\tilde{X}\cdot(f\circ g)\right] = X_p(\tilde{X}\cdot f) \\ =& (d^2f)_p(X,X),\end{aligned} \tag{2.4.2}$$

where $\tilde{X}$ is a smooth vector field on M such that at the point of p it coincides with X_p. By (2.4.1) and (2.4.2),

$$(d^2f)_{g\cdot p}(g\cdot X, g\cdot X) = (d^2f)_p(X,X). \tag{2.4.3}$$

Therefore, g preserves the index of critical manifolds, i.e., the critical manifolds on the orbit $G \cdot B$ have the same index. Now the orbit $G \cdot B$ is a G-invariant submanifold of M, which we still denote by B when there is no confusion involved.

By equivariant Morse-Bott Lemma [51], we know that B possesses a G-invariant tubular neighborhood (h, N) such that:
(1) N is a G-vector bundle over B, which is endowed with G-invariant scalar product g^N. Moreover N, which has rank $m-n$, splits into two orthogonal G-subbundles $N = N^- \oplus N^+$, where the rank of N^- is n^- and the rank of N^+ is n^+.
(2) h G-embeds N into M. Moreover there is an open G-invariant neighborhood $\mathcal{B}$ of B in N such that if $Z = (Z^-, Z^+) \in \mathcal{B}$, then

$$f(h(Z)) = c - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \tag{2.4.4}$$

where c denotes the constant values $f(B)$.

We choose a G-invariant metric g^{TB} on critical manifold B. Then we get a G-invariant metric on the total space N, namely $g^{TN} = \pi^* g^{TB} \oplus g^N$. We can obtain a G-invariant metric on M such that it has the form $g^{TM} = \pi^* g^{TB} \oplus g^N$ near the critical manifold B. Then G commutes with the Levi-Civita connection ∇^{TB} on B and Euclidean connection ∇^N on the bundle N and thus G commute with ∇^{TN} on the tangent bundle TN. Applying the result in Section 2.3, we get the following analogue of Proposition 2.8.

Proposition 2.32. *There exist $C_0 > 0, T_0 > 0$ such that when $T > T_0$, the number of eigenvalues in $[0, C_0)$ of $D_T^2|_{\Omega^j(M)}$ equals to q_j. Moreover, the eigenspaces are all G-space.*

Proof. For any $g \in G$, g commutes with the deformed de-Rham operator D_T. Thus the eigenspaces of D_T are all G-space. The rest is exactly the same as in Section 2.3. The proof of Proposition 2.32 is complete. □

We now use the notations from Section 2.3. Let $\mathrm{F}_{T,j}^{C_0}$ denotes the q_j-dimensional vector space generated by the eigenspaces of $D_T^2|_{\Omega^j(M)}$ associated with the eigenvalues in $[0, C_0), j = 0, 1, \cdots, m$ as in Section 2.3. From Proposition 2.32, G maps $\mathrm{F}_{T,j}^{C_0}$ into $\mathrm{F}_{T,j}^{C_0}$.

Let $P_T^{C_0}$ be the orthogonal projection from E^0 to $\mathrm{F}_T^{C_0}$ with $\mathrm{F}_{T,j}^{C_0} = P_j \mathrm{F}_T^{C_0}$. The isometric map $J_T : \mathrm{F}^0 \to \mathrm{E}^0$ is given as in (2.3.89). Let $e_T : \mathrm{F}^0 \to \mathrm{F}_T^{C_0}$ be defined by

$$e_T = P_T^{C_0} J_T. \tag{2.4.5}$$

We now show that e_T is an G-isomorphism from F_j onto its image when T is large enough.

Lemma 2.33. *There exists $C > 0$ such that as $T \to +\infty$, for any $s \in F_j$,*

$$(e_T - J_T)s = O(\frac{C}{\sqrt{T}})\|s\|_{F^0} \quad \text{uniformly on } M. \tag{2.4.6}$$

In particular, e_T is an G-isomorphism from G-space F_j onto G-space $\mathrm{F}_{T,j}^{C_0}$.

Proof. It is clear that e_T maps F_j into $\mathrm{F}_{T,j}^{C_0}$ and

$$(e_T - J_T)s = p_T P_T^{C_0} J_T s - J_T s + p_T^{\perp} P_T^{C_0} J_T s. \tag{2.4.7}$$

By (2.3.202), (2.3.208), (2.3.209) and (2.3.220), for any $s \in F_j$,

$$\begin{aligned}\big\|(e_T - J_T)s\big\|_{\mathrm{E}^0} \leqslant & \big\|\big(P_T^{C_0}\big)_1 J_T s - J_T s\big\|_{\mathrm{E}^0} + \big\|\big(P_T^{C_0}\big)_3 J_T s\big\|_{\mathrm{E}^0} \\ \leqslant & \big\|J_T^{-1}\big(P_T^{C_0}\big)_1 J_T - Q\big\|_{\infty} \cdot \big\|s\big\|_{\mathrm{F}^0} + \big\|\big(P_T^{C_0}\big)_3\big\|_{\infty} \cdot \big\|s\big\|_{\mathrm{F}^0} \\ \leqslant & \frac{C}{\sqrt{T}}\big\|s\big\|_{\mathrm{F}^0}. \end{aligned} \tag{2.4.8}$$

Then

$$\big\|e_T s\big\|_{\mathrm{E}^0} \geqslant \big\|J_T s\big\|_{\mathrm{E}^0} - \big\|(e_T - J_T)s\big\|_{\mathrm{E}^0} \geqslant (1 - \frac{C}{\sqrt{T}})\big\|s\big\|_{\mathrm{E}^0}. \tag{2.4.9}$$

Therefore, e_T is injective on subspace F_j when T is large enough. Moreover,

$$\dim\, F_j = \dim\, F_{T,j}^{C_0} = q_j. \tag{2.4.10}$$

Thus e_T is an isomorphism from F_j onto $\mathrm{F}_{T,j}^{C_0}$.

Since g^N is G-invariant,

$$|g^{-1} \cdot Z|_{g^N_{g^{-1}\cdot y}} = |Z|_{g^N_y}, \quad g \cdot \theta = \theta. \tag{2.4.11}$$

From (2.3.87) we have $\alpha_T(y) = \alpha_T(g^{-1} \cdot y)$. Then by (2.3.89) for any $s \in \mathrm{F}^0$,

$$\begin{aligned}\big(g \cdot J_T s\big)(y, Z) =& g \cdot (J_T s)(g^{-1} \cdot y, g^{-1} \cdot Z) \\ =& \frac{1}{\sqrt{\alpha_T(g^{-1} \cdot y)}}\rho\Big(|g^{-1} \cdot Z|_{g^N_{g^{-1}\cdot y}}\Big)\exp\Big(- \frac{T|g^{-1} \cdot Z|^2_{g^N_{g^{-1}\cdot y}}}{2}\Big) \times \\ & g \cdot s(g^{-1} \cdot y) \wedge g \cdot \theta_{g^{-1}\cdot y} \\ =& \frac{1}{\sqrt{\alpha_T(y)}}\rho\Big(|Z|_{g^N_y}\Big)\exp\Big(- \frac{T|Z|^2_{g^N_y}}{2}\Big)\big(g \cdot s\big)(y) \wedge (g \cdot \theta)_y \\ =& J_T\big(g \cdot s\big)(y, Z). \end{aligned} \tag{2.4.12}$$

That is, g commutes with J_T. Since g commutes with D_T, then g commutes with $P_T^{C_0}$ by (2.3.222). Therefore, e_T is a G map, i.e., it commute with the action of G. The proof of Lemma 2.33 is complete. □

Proof of Theorem 2.1. Recall that $V^1, \cdots, V^{l_0}$ are the irreducible representations of G. As G-representation space, $F_{T,j}^{C_0}$ can be decomposed as follows:

$$F_{T,j}^{C_0} = \sum_{\alpha=1}^{l_0} \mathrm{Hom}_G\big(V^{\alpha}, F_{T,j}^{C_0}\big) \otimes V^{\alpha}. \tag{2.4.13}$$

Then $\big(\mathrm{Hom}_G(V^\alpha, F^{C_0}_{T,j}) \otimes V^\alpha, d_T\big)$ is a G-subcomplex of the complex $(\mathrm{F}^{C_0}_{T,j}, d_T)$. From Lemma 2.33, we find that

$$\dim\, \mathrm{Hom}_G(V^\alpha, F^{C_0}_{T,j}) = d^\alpha_j. \tag{2.4.14}$$

From the Hodge theorem for finite-dimensional vector space, we know that the j-th cohomology group of the complex $(F^{C_0}_{T,\cdot}, d_T)$ is equal to the dimension of Ker $D^2_T|_{\Omega^j(M)}$.By the following Lemma 2.34, the dimension of the j-th cohomology group associated to the complex $(\mathrm{Hom}_G(V^\alpha, F^{C_0}_{T,\cdot}) \otimes V^\alpha, d_T)$ equals $b^\alpha_j \cdot \dim V^\alpha$. Now (2.1.12) and (2.1.13) follow from standard algebraic argument ([30, Lemma 3.2.12]). The proof of our main Theorem 2.1 is complete. □

Lemma 2.34.

$$H^j\Big(\mathrm{Hom}_G\big(V^\alpha, \mathrm{F}^{C_0}_{T,\cdot}\big), d_T\Big) \simeq \mathrm{Hom}_G\Big(V^\alpha, H^j\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big). \tag{2.4.15}$$

Proof. Set

$$T = \mathrm{Hom}_G\Big(V^\alpha, \cdot\Big). \tag{2.4.16}$$

Then T is an exact factor, i.e., for any exact sequence

$$E \to F \to G, \tag{2.4.17}$$

the following sequence is exact:

$$T\big(E\big) \to T\big(F\big) \to T\big(G\big). \tag{2.4.18}$$

We need to show

$$H^j\Big(T\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big) \simeq T\Big(H^j\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big). \tag{2.4.19}$$

Set

$$A_j = \mathrm{Im}\Big(\mathrm{F}^{C_0}_{T,j-1} \to \mathrm{F}^{C_0}_{T,j}\Big),\ B_j = \mathrm{Ker}\Big(\mathrm{F}^{C_0}_{T,j} \to \mathrm{F}^{C_0}_{T,j+1}\Big). \tag{2.4.20}$$

Then the following sequence is exact:

$$0 \to A_j \to B_j \to H^j\big(\mathrm{F}^{C_0}_{T,\cdot}\big) \to 0. \tag{2.4.21}$$

Since T is exact, the following sequence is also exact:

$$0 \to T\big(A_j\big) \to T\big(B_j\big) \to T\Big(H^j\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big) \to 0. \tag{2.4.22}$$

By definition,

$$H^j\Big(T\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big) = \frac{\mathrm{Ker}\Big(T\big(\mathrm{F}^{C_0}_{T,j}\big) \to T\big(\mathrm{F}^{C_0}_{T,j+1}\big)\Big)}{\mathrm{Im}\Big(T\big(\mathrm{F}^{C_0}_{T,j-1}\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big)\Big)}. \tag{2.4.23}$$

It is clear that

$$B_j \to \mathrm{F}^{C_0}_{T,j} \to \mathrm{F}^{C_0}_{T,j+1} \tag{2.4.24}$$

is exact, where $i : B_j \to \mathrm{F}^{C_0}_{T,\cdot}$ is the inclusion, then

$$T\big(B_j\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big) \to T\big(\mathrm{F}^{C_0}_{T,j+1}\big) \tag{2.4.25}$$

is also exact. Therefore,

$$\mathrm{Ker}\Big(T\big(\mathrm{F}^{C_0}_{T,j}\big) \to T\big(\mathrm{F}^{C_0}_{T,j+1}\big)\Big) = \mathrm{Im}\Big(T\big(B_j\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big)\Big). \tag{2.4.26}$$

It is clear that in the sequence

$$\mathrm{F}^{C_0}_{T,j-1} \to A_j \to \mathrm{F}^{C_0}_{T,j}, \tag{2.4.27}$$

the first map is surjective and the second inclusion map is injective. Then for sequence

$$T\big(\mathrm{F}^{C_0}_{T,j-1}\big) \to T\big(A_j\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big), \tag{2.4.28}$$

the first map is surjective and the second is injective. Therefore,

$$\mathrm{Im}\Big(T\big(\mathrm{F}^{C_0}_{T,j-1}\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big)\Big) = \mathrm{Im}\Big(T\big(A_j\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big)\Big). \tag{2.4.29}$$

From (2.4.23), (2.4.26) and (2.4.29), we get

$$H^j\Big(T\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big) = \frac{\mathrm{Im}\Big(T\big(B_j\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big)\Big)}{\mathrm{Im}\Big(T\big(A_j\big) \to T\big(\mathrm{F}^{C_0}_{T,j}\big)\Big)}. \tag{2.4.30}$$

Consider the surjective map:

$$\phi : T\big(B_j\big) \to H^j\Big(T\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big). \tag{2.4.31}$$

It is clear that

$$\mathrm{Ker}(\phi) = \mathrm{Im}\Big(T\big(A_j\big) \to T\big(B_j\big)\Big). \tag{2.4.32}$$

Hence, we get the following exact sequence:

$$0 \to T\big(A_j\big) \to T\big(B_j\big) \to H^j\Big(T\big(\mathrm{F}^{C_0}_{T,\cdot}\big)\Big) \to 0. \tag{2.4.33}$$

Combining (2.4.22) and (2.4.33), we get (2.4.19). The proof of Lemma 2.34 is complete. □

2.5 Application of equivariant Morse inequalities

In this Section, we apply the equivariant Morse inequalities (2.1.12) and (2.1.13) to prove Theorem 2.2, which is a generalization of [54, Theorem 1].

We first recall the settings for our application of equivalent Morse inequalities.

Let M be a smooth m-dimensional oriented, connected manifold with nonempty boundary ∂M. Let $f : M \to \mathbb{R}$ be a smooth function such that it is a Morse-Bott function in the interior of M. Moreover, we assume that the following condition holds. Let $\partial M = N_+ \sqcup N_-$ be a disjoint union of closed manifolds such that $f(y,u) = \frac{1}{2}u^2 + f_+(y)$ in collar neighborhood of $N_+ \times [0,1)$, while $f(y,u) = -\frac{1}{2}u^2 + f_-(y)$ in collar neighborhood of $N_- \times [0,1)$, here f_+ (resp. f_-) is a Morse-Bott function on N_+ (resp. N_-).

Let $N_+ = N_{a+} \sqcup N_{r+}$ and $N_- = N_{a-} \sqcup N_{r-}$ be disjoint union of closed manifolds. The subscripts "a" and "r" refer respectively to absolute and relative boundary condition. Let w be a smooth differential j-form on M. In a collar neighborhood U of ∂M it takes the form

$$w|_U = w_1 + du \wedge w_2, \tag{2.5.1}$$

where w_1, w_2 are u-depending differential forms not containing the factor du. Differential form w satisfies the relative boundary condition N_r if

$$w_1(y,0) = 0, \quad \text{and} \quad \frac{\partial w_2}{\partial u}(y,0) = 0. \tag{2.5.2}$$

Differential form w satisfies the absolute boundary condition N_a if

$$\frac{\partial w_1}{\partial u}(y,0) = 0, \quad \text{and} \quad w_2(y,0) = 0. \tag{2.5.3}$$

From now on we impose the relative boundary condition on N_r and the absolute boundary condition on N_a.

In the sequel, we assume collar neighborhood $U = \partial M \times [0,1)$ just for convenience. Since Theorem 2.2 is a topological result independent of our choice of metric on M, we choose a metric g^{TM} such that in collar neighborhood $\partial M \times [0,1)$, it takes the product form $g^{TM} = g^{T(\partial M)} \oplus d^2u$, where $g^{T(\partial M)}$ is a Riemannian metric on ∂M.

2.5.1 Interpretation of the boundary conditions

In this subsection, we give another interpretation of the above two boundary conditions.

Given the Riemannian metric g^{TM} on M, every smooth j-form w has in every point of the boundary a natural decomposition into the norm and the tangential component:

$$w = w_{\tan} + w_{\text{norm}}. \tag{2.5.4}$$

For $w \in \Omega^j(M)$, we consider the following boundary conditions ,

$$w_{\tan} = 0, \quad (\delta w)_{\tan} = 0 \quad \text{ on } N_r, \tag{2.5.5}$$

and

$$w_{\text{norm}=0}, \quad (dw)_{\text{norm}} = 0 \ \text{ on } N_a. \tag{2.5.6}$$

Then we have

Lemma 2.35. *The boundary condition* (2.5.5), (2.5.6) *is equivalent to the relative boundary condition* (2.5.2) *and absolute boundary condition* (2.5.3), *respectively.*

Proof. In the collar neighborhood $\partial M \times [0, 1)$, let

$$w(y, u) = w_1(y, u) + du \wedge w_2(y, u), \ \ y \in \partial M, u \in [0, 1), \tag{2.5.7}$$

where w_1 and w_2 are u-depending differential forms which do not contain the factor du. The decomposition (2.5.7) does depend on the coordinate system chosen in the product neighborhood. Then

$$w_{\tan} = w_1(y, 0), \quad w_{\text{norm}} = du \wedge w_2(y, 0). \tag{2.5.8}$$

Let ∇^{TM} be the Levi-Civita connection and $\nabla^{T(\partial M)}$ be the Levi-Civita connection on ∂M induced by ∇^{TM}. Then $\nabla^{T(\partial M)}$ induces an Euclidean connection on $\Lambda T^*(\partial M)$, which we denote by $\nabla^{\Lambda T^*(\partial M)}$. Since the metric g^{TM} takes the product form over $\partial M \times [0, 1)$, the boundary ∂M is totally geodesic in M. Let $e_1, \cdots, e_{m-1}, \frac{\partial}{\partial u}$ be an orthonormal frame in the product neighborhood $\partial M \times [0, 1)$. Then

$$d = \sum_{i=1}^{m-1} e^i \wedge \nabla_{e_i}^{\Lambda T^*(\partial M)} + du \wedge \frac{\partial}{\partial u}, \tag{2.5.9}$$

and

$$\delta = -\sum_{i=1}^{m-1} i_{e_j} \nabla_{e_j}^{\Lambda T^*(\partial M)} - i_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}. \tag{2.5.10}$$

Therefore in the product neighborhood $\partial M \times [0, 1)$,

$$dw = \sum_{j=1}^{m-1} e^j \wedge \nabla_{e_j}^{\Lambda T^*(\partial M)} w_1 + du \wedge \Big(\frac{\partial w_1}{\partial u} - \sum_{j=1}^{m-1} e^j \wedge \nabla_{e_j}^{\Lambda T^*(\partial M)} w_2 \Big), \tag{2.5.11}$$

and

$$\delta w = -\frac{\partial w_2}{\partial u} - \sum_{j=1}^{m-1} i_{e_j} \wedge \nabla_{e_j}^{\Lambda T^*(\partial M)} w_1 + du \wedge \sum_{j=1}^{m-1} i_{e_j} \nabla_{e_j}^{\Lambda T^*(\partial M)} w_2. \tag{2.5.12}$$

By (2.5.11),

$$(dw)_{\tan} = \sum_{j=1}^{m-1} e^j \wedge \nabla_{e_j}^{\Lambda T^*(\partial M)} w_1(y, 0), \tag{2.5.13}$$

and

$$(dw)_{\text{norm}} = du \wedge \Big[\frac{\partial w_1}{\partial u}(y,0) - \sum_{j=1}^{m-1} e^j \wedge \nabla_{e_j}^{\Lambda T^*(\partial M)} w_2(y,0)\Big]. \tag{2.5.14}$$

From (2.5.12),

$$(\delta w)_{\tan} = -\frac{\partial w_2}{\partial u}(y,0) - \sum_{j=1}^{m-1} i_{e_j} \wedge \nabla_{e_j}^{\Lambda T^*(\partial M)} w_1(y,0), \tag{2.5.15}$$

and

$$(\delta w)_{\text{norm}} = du \wedge \sum_{j=1}^{m-1} i_{e_j} \nabla_{e_j}^{\Lambda T^*(\partial M)} w_2(y,0). \tag{2.5.16}$$

Now Lemma 2.35 follows from (2.5.2), (2.5.3), (2.5.8), (2.5.14) and (2.5.15). From (2.5.8), (2.5.13) and (2.5.16), it is obvious that $w_{\tan} = 0$ implies $(dw)_{\tan} = 0$ and $w_{\text{norm}} = 0$ implies $(\delta w)_{\text{norm}} = 0$. □

2.5.2 Hodge theorem for manifolds with boundary

In this subsection, we will briefly introduce Hodge theorem (Theorem 2.36) for manifolds with nonempty boundary endowed with mixed boundary conditions.

Consider the following boundary conditions for $w \in \Omega^j(M)$:

$$\begin{aligned} &w_{\tan} = 0,\ (\delta w)_{\tan} = 0,\ \text{on } N_r;\\ &w_{\text{norm}} = 0, (dw)_{\text{norm}} = 0,\ \text{on } N_a. \end{aligned} \tag{2.5.17}$$

Let $\Omega_2^j(M) \subset \Omega^j(M)$ be the subspace consisting of all smooth forms which satisfy the boundary conditions (2.5.17).

Recall D^M is defined in (2.3.43). Set $\Delta = (D^M)^2$. For any $w_1, w_2 \in \Omega^j(M)$, we have ([48, (10.19)]),

$$\begin{aligned} \big\langle \Delta w_1, w_2 \big\rangle =& \big\langle dw_1, dw_2 \big\rangle + \big\langle \delta w_1, \delta w_2 \big\rangle \\ &+ \int_{\partial M} \Big[\big\langle e^n \wedge \delta w_1, w_2 \big\rangle - \big\langle i_{e_n} dw_1, w_2 \big\rangle\Big] dv_{\partial M}. \end{aligned} \tag{2.5.18}$$

By taking adjoint ([48, (10.20)]),

$$\begin{aligned} \big\langle \Delta w_1, w_2 \big\rangle =& \big\langle dw_1, dw_2 \big\rangle + \big\langle \delta w_1, \delta w_2 \big\rangle \\ &+ \int_{\partial M} \Big[\big\langle \delta w_1, i_{e_n} w_2 \big\rangle - \big\langle dw_1, e^n \wedge w_2 \big\rangle\Big] dv_{\partial M}. \end{aligned} \tag{2.5.19}$$

Here e_n is the outward-pointing unit normal to ∂M, e^n is the dual of e_n with respect to the product metric near ∂M and $dv_{\partial M}$ is the volume form on ∂M induced by the volume form on M. See [48] for more details.

In our notation, we find that

$$\begin{aligned}\big\langle \Delta w_1, w_2\big\rangle =&\big\langle dw_1, dw_2\big\rangle + \big\langle \delta w_1, \delta w_2\big\rangle + \\ &\int_{\partial M}\Big[\big\langle e^n \wedge (\delta w_1)_{\tan}, (w_2)_{\mathrm{norm}}\big\rangle - \big\langle i_{e_n}(dw_1)_{\mathrm{norm}}, (w_2)_{\tan}\big\rangle\Big] dv_{\partial M}\end{aligned} \quad (2.5.20)$$

and that

$$\begin{aligned}\big\langle \Delta w_1, w_2\big\rangle =&\big\langle dw_1, dw_2\big\rangle + \big\langle \delta w_1, \delta w_2\big\rangle + \\ &\int_{\partial M}\Big[\big\langle (\delta w_1)_{\tan}, i_{e_n}(w_2)_{\mathrm{norm}}\big\rangle - \big\langle (dw_1)_{\mathrm{norm}}, e^n \wedge (w_2)_{\tan}\big\rangle\Big] dv_{\partial M}.\end{aligned} \quad (2.5.21)$$

From (2.5.20) and (2.5.21), we deduce that for $w_1, w_2 \in \Omega_2^j(M)$, the following relation holds:

$$\big\langle \Delta w_1, w_2\big\rangle = \big\langle dw_1, dw_2\big\rangle + \big\langle \delta w_1, \delta w_2\big\rangle. \quad (2.5.22)$$

Let $\mathcal{H}^j(M, N_r)$ be the space of harmonic fields, i.e., $w \in \mathcal{H}^j(M, N_r)$ if and only if $w \in \Omega_2^j(M)$ and $dw = 0, \delta w = 0$. Let $H^j(M, N_r)$ be the j-th cohomology group associated to the complex $(\Omega^j(M, N_r), d)$, where $\Omega^j(M, N_r)$ consists of $w \in \Omega^j(M)$ such that $w_{\tan} = 0$ on N_r.

The following result is established in [40, Corollary 5.7] ([36, Section 1]).

Theorem 2.36. *The following isomorphism holds:*

$$\mathcal{H}^j(M, N_r) \simeq H^j(M, N_r). \quad (2.5.23)$$

Let $\hat{M}$ be the doubling manifold of M, i.e., $\hat{M} = M \cup_{\partial M} (-M)$ with $(-M)$ another copy of M. Let τ be the involution of $\hat{M}$ which interchanges M and $(-M)$ and leaves ∂M fixed and $i : M \to \hat{M}$ be the inclusion.

Lemma 2.37. *If $w \in \Omega^j(\hat{M})$ satisfies $\tau^* w = w$, then*

$$w_{norm} = 0, \quad (dw)_{norm} = 0 \quad \text{on } \partial M. \quad (2.5.24)$$

If $w \in \Omega^j(\hat{M})$ satisfies $\tau^* w = -w$, then

$$w_{tan} = 0, \quad (\delta w)_{tan} = 0 \quad \text{on } \partial M. \quad (2.5.25)$$

Here the meanings of w_{tan} and w_{norm} are clear (we may interpret them as the meanings for $i^* w \in \Omega^j(M)$).

Proof. We assume that in product neighborhood $\partial M \times (-1, 1)$,

$$w(y, u) = \psi_1(y, u) + du \wedge \psi_2(y, u), \quad (2.5.26)$$

where ψ_1 and ψ_1 are u-depending differential forms not containing the factor du. Then

$$w_{\tan} = \psi_1(y, 0), \quad w_{\mathrm{norm}} = du \wedge \psi_2(y, 0). \quad (2.5.27)$$

Since τ acts as identity on the boundary ∂M,

$$\tau_* = \mathrm{Id} \text{ on } T(\partial M). \tag{2.5.28}$$

Moreover,

$$\tau_*(\frac{\partial}{\partial u}) = -\frac{\partial}{\partial u}. \tag{2.5.29}$$

From (2.5.28) and (2.5.29), we get

$$(\tau^* w)(y,u) = (\tau^*\psi_1)(y,u) - du \wedge (\tau^*\psi_2)(y,u) = \psi_1(y,u) - du \wedge \psi_2(y,u). \tag{2.5.30}$$

Therefore,

$$(\tau^* w)_{\tan} = \psi_1(y,0), \quad (\tau^* w)_{\mathrm{norm}} = -du \wedge \psi_2(y,0). \tag{2.5.31}$$

From (2.5.27) and (2.5.31), $\tau^* w = w$ implies $\psi_2(y,0) = 0$ and $\tau^* w = -w$ gives $\phi_1(y,0) = 0$. Since $\tau^* w = w$ implies $\tau^*(dw) = dw$, (2.5.24) holds. And $\tau^* w = -w$ gives $\tau^*\delta w = -\delta w$, then (2.5.25) holds. The proof of Lemma 2.37 is complete. □

The doubling manifold $\hat{M}$ carries a natural $\mathbb{Z}_2$ action. Let g be the nontrivial element in $\mathbb{Z}_2$, then $gx = \tau(x)$ for any $x \in \hat{M}$. For $w \in \Omega^j(\hat{M})$, the condition $\tau^* w = w$ (resp. $\tau^* w = -w$) is equivalent to $gw = w$ (resp. $gw = -w$).

Set

$$\mathcal{H}^j(\hat{M}) = \{w \in \Omega^j(\hat{M}), (D^{\hat{M}})^2 w = 0\}. \tag{2.5.32}$$

Set

$$\begin{aligned} \mathcal{H}^j_+(\hat{M}) =& \big\{w \in \mathcal{H}^j(\hat{M}) \big|\ \tau^* w = w\big\}, \\ \mathcal{H}^j_-(\hat{M}) =& \big\{w \in \mathcal{H}^j(\hat{M}) \big|\ \tau^* w = -w\big\}. \end{aligned} \tag{2.5.33}$$

Then

$$\mathcal{H}^j(\hat{M}) = \mathcal{H}^j_+(\hat{M}) \oplus \mathcal{H}^j_-(\hat{M}) \tag{2.5.34}$$

is exactly the decomposition as $\mathbb{Z}_2$-representation spaces.

We now state an important result from [28, Proposition 1.27]:

Theorem 2.38. *The following relations hold:*

$$\begin{aligned} \dim\, \mathcal{H}^j_+(\hat{M}) =& \beta_j(M), \\ \dim\, \mathcal{H}^j_-(\hat{M}) =& \beta_j(M, \partial M). \end{aligned} \tag{2.5.35}$$

2.5.3 Special case: $f|_{\partial M} = 0$

In this subsection, we prove Theorem 2.2 in a special case, namely $f\big|_{\partial M} = 0$ and all critical points of f in the interior of M are isolated and non-degenerate. Then $f(y,u) = \frac{u^2}{2}$ in $N_+ \times [0,1)$ and $f(y,u) = -\frac{u^2}{2}$ in $N_- \times [0,1)$.

Set $N_a = N_{a+} \sqcup N_{a-}, N_r = N_{r+} \sqcup N_{r-}$. Set

$$\begin{aligned}\beta_j(M, N_r) &= \dim H^j(M, N_r),\\ \beta_j(N_{r-}) &= \dim H^j(N_{r-}),\\ \beta_j(N_{a+}) &= \dim H^j(N_{a+}).\end{aligned} \tag{2.5.36}$$

We denote by c_j the number of non-degenerate critical points with Morse index j. Then Theorem 2.2 reduces to the following forms.

Theorem 2.39. *The following inequalities hold for* $0 \leqslant k \leqslant m$

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, N_r) \leqslant \sum_{j=0}^{k}(-1)^{k-j}\upsilon_j, \tag{2.5.37}$$

where

$$\upsilon_j = c_j + \beta_j(N_{a+}) + \beta_{j-1}(N_{r-}). \tag{2.5.38}$$

The equality holds for k=m.

The function f may be easily extend to the whole manifold $\hat{M}$ via the $\mathbb{Z}_2$ action. More precisely, set

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in M,\\ f(-x) & \text{if } x \in -M,\end{cases}$$

where $-x$ denotes the point $g \cdot x$. We also denote by f the extended function on $\hat{M}$ when there is no confusion.

Set $b = \sum_{j=0}^{m} c_j$. Let $\{x_1, \cdots, x_b\}$ be the isolated non-degenerate critical points for f in M, then the isolated non-degenerate critical points in $\hat{M}$ for the extended function are $\{x_1, \cdots, x_b, -x_1, \cdots, -x_b\}$ and the Morse index for any x_k, $1 \leqslant k \leqslant m$, is unchanged. Also ∂M is a critical submanifold of f in the sense of Bott [8].

Denote by $\mathbb{R}^+, \mathbb{R}^-$ the trivial and the nontrivial one dimensional real $\mathbb{Z}_2-$representatation respectively.

Proof of Theorem 2.39. We divide the proof of inequalities (2.5.37) into three cases.
Case 1. $\partial M = N_+$, i.e., $N_- = \emptyset$.

Under the condition that $\partial M = N_+$, the inequalities (2.5.37) turn to the following form:

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, N_{r+}) \leqslant \sum_{j=0}^{k}(-1)^{k-j}[c_j + \beta_j(N_{a+})]. \tag{2.5.39}$$

Moreover, the equality holds when $k = m$, i.e.,

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M, N_{r+}) = \sum_{j=0}^{m}(-1)^{m-j}\big[c_j + \beta_j(N_{a+})\big]. \tag{2.5.40}$$

From Theorem 2.38,

$$b_j^1 = \beta_j(M), \quad b_j^2 = \beta_j(M, \partial M). \tag{2.5.41}$$

On the other hand, the general expression (2.1.10) may be interpreted as:

$$d_j^1 = \beta_j(\partial M) + c_j \quad d_j^2 = c_j. \tag{2.5.42}$$

We now prove (2.5.42). For critical point x of f and $x \in M\backslash\partial M$, set

$$W = \big\{x\big\} \oplus \big\{\tau(x)\big\}. \tag{2.5.43}$$

Here $\big\{x\big\}$ is the real line generated by x. Then W is a 2-dimensional real vector space spanned by x and $\tau(x)$ and $\mathbb{Z}_2$ acts naturally on W. W can be rewritten as

$$W = \big\{x + \tau(x)\big\} \oplus \big\{x - \tau(x)\big\}. \tag{2.5.44}$$

Moreover, the 1-dimensional space $\big\{x + \tau(x)\big\}$ (resp. $\big\{x - \tau(x)\big\}$) is isomorphic to $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) as $\mathbb{Z}_2-$representation spaces. Therefore, as a $\mathbb{Z}_2-$representation space, W can be decomposed as

$$W = \mathbb{R}^+ \oplus \mathbb{R}^-. \tag{2.5.45}$$

Hence (2.5.42) holds.

From (2.5.41) and (2.5.42), the equivariant Morse equalities (2.1.12) should be read now

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M) \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big(\beta_j(\partial M) + c_j\big), \tag{2.5.46}$$

and

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, \partial M) \leqslant \sum_{j=0}^{k}(-1)^{k-j}c_j. \tag{2.5.47}$$

The equalities hold in (2.5.46) and (2.5.47) when $k = m$.

As $\partial M = N_{r+} \sqcup N_{a+}$ is disjoint union,

$$H^j(\partial M, N_{r+}) \simeq H^j(N_{a+}). \tag{2.5.48}$$

Moreover, it is obviously that

$$H^0(M, \partial M) = 0. \tag{2.5.49}$$

Consider the Mayer-Vietoris sequence ([33, p. 185]) associated to the triad $(M, \partial M, N_{r+})$:

$$\cdots \to H^{j-1}(N_{a+}) \to H^j(M, \partial M) \to H^j(M, N_{r+}) \to H^j(N_{a+}) \to \cdots , \tag{2.5.50}$$

which begins as

$$0 \to H^0(N_{a+}) \to H^1(M, \partial M) \to \cdots , \tag{2.5.51}$$

and ends as

$$\cdots \to H^{m-1}(N_{a+}) \to H^m(M, \partial M) \to H^m(M, N_{r+}) \to 0. \tag{2.5.52}$$

For the sequence (2.5.50), by standard algebraic technique ([30, Lemma 3.2.12]), we find that for $k = 0, 1, \cdots, m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\big[\beta_j(N_{a+}) - \beta_j(M, N_{r+}) + \beta_j(M, \partial M)\big] = \dim \operatorname{Im} \delta^k, \tag{2.5.53}$$

where

$$\delta^k : H^k(N_{a+}) \to H^{k+1}(M, \partial M) \tag{2.5.54}$$

is the connecting morphism in the long exact sequence (2.5.50). Note

$$\dim \operatorname{Im} \delta^m = 0. \tag{2.5.55}$$

From (2.5.47), we get

$$\sum_{j=0}^{k}(-1)^{k-j}\big[\beta_j(M, \partial M) + \beta_j(N_{a+})\big] \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big[c_j + \beta_j(N_{a+})\big]. \tag{2.5.56}$$

The equality in (2.5.56) holds when $k = m$. Combining (2.5.53), (2.5.55) and (2.5.56), we get (2.5.39) and (2.5.40). The proof of Case 1 is complete.
Case 2. $\partial M = N_-$, i.e., $N_+ = \emptyset$.

The inequalities (2.5.37) turn to

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, N_{r-}) \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big[c_j + \beta_{j-1}(N_{r-})\big]. \tag{2.5.57}$$

And the equality holds when $k = m$, i.e.,

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M, N_{r-}) = \sum_{j=0}^{m}(-1)^{m-j}\big[c_j + \beta_{j-1}(N_{r-})\big]. \tag{2.5.58}$$

In this case, b_j^1, b_j^2 are the same as in Case 1, i.e., $b_j^1 = \beta_j(M), b_j^2 = \beta_j(M, \partial M)$.

The general expression (2.1.10) may be interpreted now as:

$$d_j^1 = \beta_{j-1}(\partial M) + c_j, \quad d_j^2 = c_j.$$

Let $f_1 = -f$. Then $-f_1$ is also a Morse-Bott function. A critical point in the interior of M of Morse index j for f has Morse index $m-j$ for f_1. The critical manifold ∂M with index 1 for f has index 0 for f_1. Note the number of critical points with index j for f_1 is c_{m-j}. Now we directly apply the result (2.5.47) in the Case 1 to the Morse-Bott function f_1: for $k = 0, 1, \cdots, m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, \partial M) \leqslant \sum_{j=0}^{k}(-1)^{k-j}c_{m-j}; \tag{2.5.59}$$

when $k = m$ the equality holds:

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M, \partial M) = \sum_{j=0}^{m}(-1)^{m-j}c_{m-j}. \tag{2.5.60}$$

Poincarés duality [48, Proposition 9.12] states that:

$$H^j(M, \partial M) \simeq H^{m-j}(M). \tag{2.5.61}$$

From (2.5.60) and (2.5.61), we have

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_{m-j}(M) \leqslant \sum_{j=0}^{k}(-1)^{k-j}c_{m-j}; \tag{2.5.62}$$

when $k = m$, the equality holds:

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_{m-j}(M) = \sum_{j=0}^{m}(-1)^{m-j}c_{m-j}. \tag{2.5.63}$$

It is clear that

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_{m-j}(M) = (-1)^{k+m}\sum_{j=0}^{m}(-1)^{j}\beta_j(M) + (-1)^{k+m+1}\sum_{j=0}^{m-k-1}(-1)^{j}\beta_j(M),$$

and

$$\sum_{j=0}^{k}(-1)^{k-j}c_{m-j} = (-1)^{k+m}\sum_{j=0}^{m}(-1)^{j}c_j + (-1)^{k+m+1}\sum_{j=0}^{m-k-1}(-1)^{j}c_j. \tag{2.5.64}$$

We now obtain that for $k = 0, 1, \cdots, m-1$,

$$(-1)^{k+m+1}\sum_{j=0}^{m-k-1}(-1)^{j}\beta_j(M) \leqslant (-1)^{k+m+1}\sum_{j=0}^{m-k-1}(-1)^{j}c_j. \tag{2.5.65}$$

Then we replace k by $m-p$, for $p=1,\cdots,m$,

$$\sum_{j=0}^{p-1}(-1)^{p-1-j}\beta_j(M)\leqslant\sum_{j=0}^{p-1}(-1)^{p-1-j}c_j. \tag{2.5.66}$$

Finally we replace $p-1$ by q and we get for $q=0,1,\cdots,m-1$,

$$\sum_{j=0}^{q}(-1)^{q-j}\beta_j(M)\leqslant\sum_{j=0}^{q}(-1)^{q-j}c_j. \tag{2.5.67}$$

Note when $q=m$, the equality in (2.5.67) holds. Hence for $k=0,1,\cdots,m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M)\leqslant\sum_{j=0}^{k}(-1)^{k-j}c_j. \tag{2.5.68}$$

.

As $\partial M=N_-$, we get:

$$H^0(M,N_{r-})=0. \tag{2.5.69}$$

Consider the Mayer-Vietoris sequence ([33, p. 157]) associated to the pair (M,N_{r-}):

$$\cdots\to H^{j-1}(N_{r-})\to H^j(M,N_{r-})\to H^j(M)\to H^j(N_{r-})\to\cdots, \tag{2.5.70}$$

which begins as

$$0\to H^0(M)\to H^0(N_{r-})\to H^1(M,N_{r-})\to\cdots, \tag{2.5.71}$$

and ends as

$$\cdots\to H^{m-1}(N_{r-})\to H^m(M,N_{r-})\to H^m(M)\to 0. \tag{2.5.72}$$

From (2.5.70) and [30, Lemma 3.2.12], we have for $k=0,1,\cdots,m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\big[\beta_j(M)-\beta_j(M,N_{r-})+\beta_{j-1}(N_{r-})\big]=\dim\,\mathrm{Im}\,\sigma^k, \tag{2.5.73}$$

where σ denotes the map $H^k(M)\to H^k(N_{r-})$ in the long exact sequence induced by the inclusion map $i:N_{r-}\to M$. Note

$$\dim\,\mathrm{Im}\,\sigma^m=0. \tag{2.5.74}$$

From (2.5.68), we have

$$\sum_{j=0}^{k}(-1)^{k-j}[\beta_j(M)+\beta_{j-1}(N_{r-})]\leqslant\sum_{j=0}^{k}(-1)^{k-j}\big[c_j+\beta_{j-1}(N_{r-})\big]. \tag{2.5.75}$$

Now (2.5.57) and (2.5.58) follow from (2.5.73)−(2.5.75).
Case 3. $\partial M = N_+ \sqcup N_-, N_+ \neq \emptyset, N_- \neq \emptyset$.

We first claim the following inequalities.
For $k = 0, 1, \cdots, m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, N_+) \leqslant \sum_{j=0}^{k}(-1)^{k-j}c_j, \tag{2.5.76}$$

when $k = m$, the equality holds:

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M, N_+) = \sum_{j=0}^{m}(-1)^{m-j}c_j. \tag{2.5.77}$$

We postpone the proof of (2.5.76) and (2.5.77) later. We now prove general inequalities (2.5.37) and the associated equality via (2.5.76) and (2.5.77).
By (2.5.76), we have

$$\sum_{j=0}^{k}(-1)^{k-j}\big[\beta_j(M, N_+) + \beta_j(N_{a+})\big] \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big[c_j + \beta_j(N_{a+})\big]. \tag{2.5.78}$$

We now consider the Mayer-Vietorie sequence ([33, p. 185]) associated with the triad (M, N_+, N_{r+}):

$$\cdots \to H^{j-1}(N_{a+}) \to H^j(M, N_+) \to H^j(M, N_{r+}) \to H^j(N_{a+}) \to \cdots, \tag{2.5.79}$$

which begins as

$$0 \to H^0(N_{a+}) \to H^1(M, N_+) \to \cdots, \tag{2.5.80}$$

and ends as

$$\cdots \to H^{m-1}(N_{a+}) \to H^m(M, N_+) \to H^m(M, N_{r+}) \to 0. \tag{2.5.81}$$

From (2.5.79),

$$\sum_{j=0}^{k}(-1)^{k-j}[\beta_j(N_{a+}) - \beta_j(M, N_{r+}) + \beta_j(M, N_+)] = \dim \operatorname{Im} \delta_1^k, \tag{2.5.82}$$

where

$$\delta_1^k : H^k(N_{a+}) \to H^{k+1}(M, N_+) \tag{2.5.83}$$

is the connecting morphism in the long exact sequence (2.5.79). Note

$$\dim \operatorname{Im} \delta_1^m = 0. \tag{2.5.84}$$

From (2.5.78), (2.5.82) and (2.5.84), we get

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M,N_{r+}) + \dim \operatorname{Im} \delta_1^k \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big[c_j + \beta_j(N_{a+})\big]. \tag{2.5.85}$$

When $k = m$, the equality holds:

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M,N_{r+}) = \sum_{j=0}^{m}(-1)^{m-j}\big[c_j + \beta_j(N_{a+})\big]. \tag{2.5.86}$$

From (2.5.85) and (2.5.86), we find that

$$\sum_{j=0}^{k}(-1)^{k-j}\big[\beta_j(M,N_{r+}) + \beta_{j-1}(N_{r-})\big] + \dim \operatorname{Im} \delta_1^k \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big[c_j + \beta_j(N_{a+}) + \beta_{j-1}(N_{r-})\big]. \tag{2.5.87}$$

When $k = m$,

$$\sum_{j=0}^{m}(-1)^{m-j}\big[\beta_j(M,N_{r+}) + \beta_{j-1}(N_{r-})\big] = \sum_{j=0}^{m}(-1)^{m-j}\big[c_j + \beta_j(N_{a+}) + \beta_{j-1}(N_{r-})\big].$$

Next we consider the triad (M, N_r, N_{r+}):

$$\cdots \to H^{j-1}(N_{r-}) \to H^j(M,N_r) \to H^j(M,N_{r+}) \to H^j(N_{r-}) \to \cdots, \tag{2.5.88}$$

which begins as

$$0 \to H^0(N_{r-}) \to H^1(M,N_r) \to \cdots, \tag{2.5.89}$$

and ends as

$$\cdots \to H^{m-1}(N_{r-}) \to H^m(M,N_r) \to H^m(M,N_{r+}) \to 0. \tag{2.5.90}$$

From (2.5.88),

$$\sum_{j=0}^{k}(-1)^{k-j}\big[\beta_j(M,N_{r+}) - \beta_j(M,N_r) + \beta_{j-1}(N_{r-})\big] = \dim \operatorname{Im} \delta_2^k, \tag{2.5.91}$$

where δ_2^k denotes the morphism $H^k(M,N_{r+}) \to H^k(N_{r-})$ in the long exact sequence (2.5.88) induced by the inclusion

$$(N_r, N_{r+}) \hookrightarrow (M, N_{r+}). \tag{2.5.92}$$

It is clear that

$$\dim \operatorname{Im} \delta_2^m = 0. \tag{2.5.93}$$

Thus from (2.5.87) and (2.5.91), we get

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M,N_r) + \sum_{i=1}^{2}\dim \operatorname{Im} \delta_i^k \leqslant \sum_{j=0}^{k}(-1)^{k-j}\big[c_j + \beta_j(N_{a+}) + \beta_{j-1}(N_{r-})\big],$$

and when $k=m$,

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M,N_r) = \sum_{j=0}^{m}(-1)^{m-j}\big[c_j + \beta_j(N_{a+}) + \beta_{j-1}(N_{r-})\big]. \tag{2.5.94}$$

The proof of (2.5.37) is complete.

Now we prove the inequalities (2.5.76) and (2.5.77).

Set

$$M_1 = M \cup_{N_+} (-M), \quad M_2 = M_1 \cup_{N'_-} (-M_1), \tag{2.5.95}$$

where N'_- is the boundary of M_1, i.e., $N'_- = N_- \sqcup (-N_-)$. From Theorem 2.38, we know that if we endow natural $\mathbb{Z}_2$ action on M_2, we would have the decomposition:

$$H^j(M_2) = H^j(M_1) \otimes_{\mathbb{R}} \mathbb{R}^+ \oplus H^j(M_1, N'_-) \otimes \mathbb{R}^-. \tag{2.5.96}$$

If we endow natural $\mathbb{Z}_2$ action on M_1, we would also have the decomposition:

$$H^j(M_1) = H^j(M) \otimes_{\mathbb{R}} \mathbb{R}^+ \oplus H^j(M, N_+) \otimes_{\mathbb{R}} \mathbb{R}^-. \tag{2.5.97}$$

The proof of (2.5.97) is similar as we do for the closed manifold, see [28, Proposition 1.27] for more details.

Now we consider natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on M_2 and apply the results for closed manifolds to the doubling manifold M_2. Let τ_1 and τ_2 be the flip maps of M_1 and M_2 respectively. Let g (resp. e) be the nontrivial (resp. trivial) element in $\mathbb{Z}_2$. $\mathbb{Z}_2$ could be viewed as a multiplication group, i.e., $g^2 = e = e^2$. Then there is a natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on M_2,

$$(e,g)\cdot x = \tau_1(x), \quad (g,e)\cdot x = \tau_2(x), \quad \forall\, x \in M_2. \tag{2.5.98}$$

Let $\{W^\alpha\}_{\alpha=1}^4$ be the non-isomorphic irreducible representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. As vector space, $W^j = \mathbb{R}$ but (e,g) acts as Id on W^1, W^2 and acts as $-$Id on W^3, W^4; besides (g,e) acts as Id on W^1, W^3 and acts as $-$Id on W^2, W^4.

From (2.5.96) and (2.5.97), we find that the general expression (2.1.11) may be interpreted as:

$$b_j^1 = \beta_j(M),\ b_j^2 = \beta_{m-j}(M),\ b_j^3 = \beta_j(M,N_+),\ b_j^4 = \beta_{m-j}(M,N_+). \tag{2.5.99}$$

Here we apply Poincare's duality again to get $\beta_j(M_1, N_1') = \beta_{m-j}(M_1)$.

The general expression (2.1.10) for Morse-Bott function $-f$ may be interpreted as :

$$\begin{aligned} &d_j^1 = c_{m-j} + \beta_{j-1}(N_+) + \beta_j(N_-),\ d_j^2 = c_{m-j} + \beta_{j-1}(N_+); \\ &d_j^3 = c_{m-j} + \beta_j(N_-),\ d_j^4 = c_{m-j}. \end{aligned} \tag{2.5.100}$$

Now we prove (2.5.100). For critical point x of f and $x \in M\backslash\partial M$, set

$$W = \{x\} \oplus \{\tau_1(x)\} \oplus \{\tau_2(x)\} \oplus \{\tau_1\tau_2(x)\}. \tag{2.5.101}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ acts naturally on W. W may be rewritten as

$$\begin{aligned} W =& \{x + \tau_1(x) + \tau_2(x) + \tau_2\tau_1(x)\} \oplus \{x + \tau_1(x) - \tau_2(x) - \tau_2\tau_1(x)\} \\ &\oplus \{x - \tau_1(x) + \tau_2(x) - \tau_2\tau_1(x)\} \oplus \{x - \tau_1(x) - \tau_2(x) + \tau_2\tau_1(x)\}. \end{aligned} \tag{2.5.102}$$

It is clear that the 1-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-representation space $\{x + \tau_1(x) + \tau_2 + \tau_2\tau_1(x)\}$ (resp. $\{x + \tau_1(x) - \tau_2(x) - \tau_2\tau_1(x)\}$, resp. $\{x - \tau_1(x) + \tau_2(x) - \tau_2\tau_1(x)\}$, resp. $\{x - \tau_1(x) - \tau_2(x) + \tau_2\tau_1(x)\}$) is isomorphic to W^1 (resp W^2, resp. W^3, resp. W^4) as $\mathbb{Z}_2 \times \mathbb{Z}_2$ space. Therefore, as a representation space of $\mathbb{Z}_2 \times \mathbb{Z}_2$, W can be decomposed as

$$W = W^1 \oplus W^2 \oplus W^3 \oplus W^4. \tag{2.5.103}$$

For non-degenerate critical manifolds, we have for any $w \in H^{j-1}(N_+)$,

$$w \oplus \tau_2(w) = \{w + \tau_2(w)\} \oplus \{w - \tau_2(w)\}; \tag{2.5.104}$$

and for any $w' \in H^j(N_-)$,

$$w' \oplus \tau_1(w') = \{w' + \tau_1(w')\} \oplus \{w' - \tau_1(w')\}. \tag{2.5.105}$$

Note that $\tau_1 = \mathrm{Id}$ on $H^{j-1}(N_+)$ and $\tau_2 = \mathrm{Id}$ on $H^j(N_-)$. Then (2.5.100) follows from (2.5.102)–(2.5.105).

Applying equivariant Morse equalities (2.1.12) to $\alpha = 4$, one can get (2.5.76) and (2.5.77) exactly the same way as we obtain (2.5.68) in Case 2. The proof of general inequalities (2.5.37) and its associated equality is complete. The proof of Theorem 2.39 is complete. □

2.5.4 Morse inequalities on manifolds with boundary

In this subsection, we prove Theorem 2.2 using equivariant Morse inequalities.

Proof of Theorem 2.2. We first establish the following relations, which plays the same role in this subsection as that of (2.5.76) and (2.5.77) in Case 3 in subsection 2.5.3. Recall q_j is given in (2.1.7). For $k = 0, 1, \cdots, m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M, N_+) \leqslant \sum_{j=0}^{k}(-1)^{k-j}q_j, \tag{2.5.106}$$

when $k = m$, the equation holds:

$$\sum_{j=0}^{m}(-1)^{m-j}\beta_j(M,N_+) = \sum_{j=0}^{m}(-1)^{m-j}q_j. \tag{2.5.107}$$

The proof of (2.5.106) and (2.5.107) is exactly the same as that of (2.5.76) and (2.5.77) in subsection 2.5.3 except that the expression (2.5.100) should be replaced by

$$\begin{aligned} d_j^1 =& q_{m-j} + \dim_{\mathbb{R}} F_{r-,j-1}(-f) + \dim_{\mathbb{R}} F_{a+,j}(-f),\\ d_j^2 =& q_{m-j} + \dim_{\mathbb{R}} F_{r-,j-1}(-f),\ d_j^3 = q_{m-j} + \dim_{\mathbb{R}} F_{a+,j}(-f),\\ d_j^4 =& q_{m-j}, \end{aligned} \tag{2.5.108}$$

where $F_j(-f)$ (resp. $F_{a+,j}(-f)$, resp. $F_{r-,j}(-f)$) denotes the corresponding vector space F_j (resp. $F_{a+,j}$, resp. $F_{r-,j}$) with respect to the Morse-Bott function $-f$.

Now we prove (2.5.108) as follows.

For every $w \in F_j(-f)$, set

$$W = \big\{w\big\} \oplus \big\{\tau_1(w)\big\} \oplus \big\{\tau_2(w)\big\} \oplus \big\{\tau_1\tau_2(w)\big\}. \tag{2.5.109}$$

Then $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts naturally on W. W can be rewritten as

$$\begin{aligned} W =& \big\{w + \tau_1(w) + \tau_2(w) + \tau_2\tau_1(w)\big\} \oplus \big\{w + \tau_1(w) - \tau_2(w) - \tau_2\tau_1(w)\big\}\\ &\oplus \big\{w - \tau_1(w) + \tau_2(w) - \tau_2\tau_1(w)\big\} \oplus \big\{w - \tau_1(w) - \tau_2(w) + \tau_2\tau_1(w)\big\}. \end{aligned} \tag{2.5.110}$$

Moreover, the 1-dimensional $\mathbb{Z}_2\times\mathbb{Z}_2$-representation space $\big\{w+\tau_1(w)+\tau_2(w)+\tau_2\tau_1(w)\big\}$ $\Big($resp. $\big\{w + \tau_1(w) - \tau_2(w) - \tau_2\tau_1(w)\big\}$, resp. $\big\{w - \tau_1(w) + \tau_2(w) - \tau_2\tau_1(w)\big\}$, resp. $\big\{w - \tau_1(w) - \tau_2(w) + \tau_2\tau_1(w)\big\}\Big)$ is isomorphic to W^1 (resp W^2, resp. W^3, resp. W^4) as representation space of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus as a representation space of $\mathbb{Z}_2 \times \mathbb{Z}_2$,

$$W \simeq W^1 \oplus W^2 \oplus W^3 \oplus W^4. \tag{2.5.111}$$

For non-degenerate critical manifolds on the boundary, we have for any $w \in F_{r-,j-1}(-f)$, $w' \in F_{a+,j}(-f)$,

$$\begin{aligned} \big\{w\big\} \oplus \big\{\tau_2(w)\big\} =& \big\{w + \tau_2(w)\big\} \oplus \big\{w - \tau_2(w)\big\};\\ \big\{w'\big\} \oplus \big\{\tau_1(w')\big\} =& \big\{w' + \tau_1(w')\big\} \oplus \big\{w' - \tau_1(w')\big\}. \end{aligned} \tag{2.5.112}$$

From (2.5.111) and (2.5.112), one get (2.5.108) immediately.

Applying equivariant Morse inequalities (2.1.12) to $\alpha = 4$, we deduce that

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_{m-j}(M,N_+) \leqslant \sum_{j=0}^{k}(-1)^{k-j}q_{m-j}. \tag{2.5.113}$$

One verifies (2.5.106) from (2.5.113) as in <u>Case 2</u> in subsection 2.5.3.

Recall that (2.5.37) follows from (2.5.76) and (2.5.77) in subsection 2.5.3. Thus from (2.5.106) and (2.5.107), we have the following analogue of (2.5.37):

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(M,N_r)\leqslant\sum_{j=0}^{k}(-1)^{k-j}\mu_j, \tag{2.5.114}$$

where

$$\mu_j=q_j+\beta_j(N_{a+})+\beta_{j-1}(N_{r-}). \tag{2.5.115}$$

The equality holds for $k=m$.

We next deduce the inequalities (2.1.16) and its associated equality from (2.5.114) and its associated equality. We directly apply our results for closed manifold to the submanifolds N_{a+} and N_{r-} respectively.

Applying degenerate Morse inequalities, i.e., (2.3.1) and (2.3.2), to closed manifold N_{a+}, we find that for $0\leqslant k\leqslant m-1$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(N_{a+})\leqslant\sum_{j=0}^{k}(-1)^{k-j}q_{a+,j}. \tag{2.5.116}$$

The equality in (2.5.116) holds for $k=m-1$. Note

$$\beta_m(N_{a+})=0=q_{a+,m}. \tag{2.5.117}$$

By (2.5.116) and (2.5.117), we get that for $0\leqslant k\leqslant m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(N_{a+})\leqslant\sum_{j=0}^{k}(-1)^{k-j}q_{a+,j}. \tag{2.5.118}$$

The equality in (2.5.118) holds when $k=m$.

For closed manifold N_{r-}, we also have for $0\leqslant k\leqslant m-1$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j(N_{r-})\leqslant\sum_{j=0}^{k}(-1)^{k-j}q_{r-,j}. \tag{2.5.119}$$

The equality holds for $k=m-1$.

From (2.5.119), we have for $0\leqslant k\leqslant m$,

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_{j-1}(N_{r-})\leqslant\sum_{j=0}^{k}(-1)^{k-j}q_{r-,j-1}. \tag{2.5.120}$$

The equality in (2.5.120) holds when $k=m$.

From (2.5.114), (2.5.118) and (2.5.120), we get (2.1.16). It is clear that its associated equality also holds when $k=m$. The proof of Theorem 2.2 is finished. □

3 The second coefficient of asymptotic expansion of Bergman kernel

The aim of this Chapter is to calculate the second coefficient of the asymptotic expansion of the Bergman kernel of the Hodge-Dolbeault operator associated to high powers of a Hermitian line bundle with non-degenerate curvature, using the method of formal power series developed by Ma and Marinescu.

This Chapter is organized as follows. In Section 3.1, we state our second main result of this Thesis, i.e., Theorem 3.3. In Section 3.2, we provide the corresponding Lichnerowicz formula for the Hodge-Dolbeault operator and the Bergman kernel. In Section 3.3, we establish the spectral gap property of the Hodge-Dolbeault operator. The spectral gap plays an essential role in the proof of the existence of the asymptotic expansion of the Bergman kernel. In Section 3.4, we investigate in great detail about the existence of the asymptotic expansion. We also provide there an explicit formula (3.4.224) of the second coefficient in the asymptotic expansion. In Section 3.5, we prove that the last terms in the formula (3.4.224) vanish. In Section 3.6, we calculate the rest terms in (3.4.224) and then finish the proof of our second main result. In Section 3.7 we check the compatibility of our final result (3.1.19) with Riemann-Roch-Hirzebruch formulas.

3.1 Main result

Let (X, J) be a compact complex manifold with complex structure J. Let (L, h^L) be a holomorphic Hermitian line bundle on X endowed with holomorphic Hermitian connection ∇^L and curvature $R^L = (\nabla^L)^2$.

Our basis assumption is that $\omega := \frac{\sqrt{-1}}{2\pi} R^L$ defines a symplectic form on X.

The complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Since the J-invariant bilinear form $\omega(\cdot, J\cdot)$ is non-degenerate on TX, there exist J-invariant subbundles denoted $V, V^{\perp} \subset TX$ such that

$$\omega(\cdot, J\cdot)\big|_V < 0, \quad \omega(\cdot, J\cdot)\big|_{V^{\perp}} > 0 \tag{3.1.1}$$

and $V, V^{\perp}$ are orthogonal with respect to $\omega(\cdot, J\cdot)$. Equivalently, there exist subbundles $W, W^{\perp} \subset T^{(1,0)}X$ such that $W \oplus W^{\perp} = T^{(1,0)}X$, $W, W^{\perp}$ orthogonal with respect to ω and

$$R^L(u, \overline{u}) < 0, \text{ for } u \in W; \; R^L(u, \overline{u}) > 0, \text{ for } u \in W^{\perp}. \tag{3.1.2}$$

Set rank$W = q$. Then the curvature R^L is non-degenerate of signature $(q, n-q)$. Now take the Riemannian metric g^{TX} on X to be

$$g^{TX} := -\omega(\cdot, J\cdot)\big|_V \oplus \omega(\cdot, J\cdot)\big|_{V^\perp}. \tag{3.1.3}$$

Since ω is compatible with the complex structure J, then the metric g^{TX} is also compatible with J. Note that (X, g^{TX}) is not necessarily Kähler. Denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induced by g^{TX}. Denote by ∇^{TX} the Levi-Civita connection on (TX, g^{TX}) and by R^{TX} (resp. r^X) its curvature (resp. scalar curvature). Set $\Lambda^{i,j} = \Lambda^i(T^{*(1,0)}X) \otimes \Lambda^j(T^{*(0,1)}X)$. Since $R^L \in \Lambda^{1,1}$, then ω is compatible with the complex structure J.

Denote by $h^{T^{(1,0)}X}$ the Hermitian metric on $T^{(1,0)}X$ induced by g^{TX}. Let $\nabla^{T^{(1,0)}X}$ be the holomorphic Hermitian connection on $(T^{(1,0)}X, h^{T^{(1,0)}X})$, and let $R^{T^{(1,0)}X}$ be its curvature. Then $\nabla^{T^{(1,0)}X}$ induces the holomorphic Hermitian connection $\nabla^{\det(T^{(1,0)}X)}$ on the holomorphic line bundle $\det(T^{(1,0)}X)$. Let ∇^{Cl} denote the Clifford connection on $\Lambda(T^{*(0,1)}X)$ induced canonically by ∇^{TX} and $\nabla^{\det(T^{(1,0)}X)}$.

Let (E, h^E) be a holomorphic Hermitian vector bundle on X endowed with holomorphic Hermitian connection ∇^E and the curvature $R^E = (\nabla^E)^2$. Set

$$E_p^j := \Lambda^j(T^{*(0,1)}X) \otimes L^p \otimes E, \quad E_p = \bigoplus_{j=1}^{n} E_p^j. \tag{3.1.4}$$

Let $\langle\cdot,\cdot\rangle$ be the metric on E_p induced by g^{TX}, h^L and h^E, and let dv_X be the Riemannian volume form of (TX, g^{TX}). Let $\Omega^{0,j}(X, L^p \otimes E)$ denote the space of smooth sections of E_p^j on X. Set $\Omega^{0,\bullet}(X, L^p \times E) = \oplus_{j=1}^n \Omega^{0,j}(X, L^p \otimes E)$. The L^2 scalar product on $\Omega^{0,\bullet}(X, L^p \otimes E)$, is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle dv_X(x). \tag{3.1.5}$$

We denote by $\|\cdot\|_{L^2}$ the corresponding norm.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ with respect to the scalar product (3.1.5).

Definition 3.1. The Hodge-Dolbeault operator on the complex manifold (X, J) is defined as

$$D_p = \sqrt{2}\big(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\big). \tag{3.1.6}$$

By the Hodge theory, we know that

$$\mathrm{Ker} D_p^2|_{\Omega^{0,j}(X, L^p\otimes E)} \simeq H^{0,j}(X, L^p \otimes E), \tag{3.1.7}$$

where $H^{0,\bullet}(X, L^p \otimes E)$ is the Dolbeault cohomology. By Andreotti-Grauert coarse vanishing theorem (see e.g. [29, (1.29)], [30, (8.2.18)]) we obtain that for p large enough,

$$H^{0,j}(X, L^p \otimes E) = 0, \quad \text{for } j \neq q. \tag{3.1.8}$$

It is a consequence of (3.1.7) and (3.1.8) that the kernel of D_p^2 is concentrated in degree q for p large enough. Let $P_p^{0,q}$ be the orthogonal projection from $\Omega^{0,q}(X, L^p \otimes E)$ on $\mathrm{Ker}(D_p^2)$, and let $P_p^{0,q}(\cdot,\cdot)$ be its kernel with respect to dv_X. The operator $P_p^{0,q}$ is smoothing, so the kernel $P_p^{0,q}(\cdot,\cdot)$ is smooth.

Let $I_{\det(\overline{W}^*)\otimes E}$ be the orthogonal projection from $\Lambda(T^{*(0,1)}X) \otimes E$ onto $\det(\overline{W}^*) \otimes E$. We denote by $\big(\det(\overline{W}^*)\big)^\perp$ the orthogonal complement of $\det(\overline{W}^*)$ in $\Lambda(T^{*(0,1)}X)$. Denote by Θ the form associated to g^{TX}, i.e.,

$$\Theta(U,V) = \big\langle JU, V\big\rangle \ \text{ for } U,V \in TX. \tag{3.1.9}$$

The following result is due to Ma and Marinescu, see [29, Theorem 1.7].

Theorem 3.2. *There exist smooth coefficients* $\mathbf{b}_r(x) \in \mathrm{End}\big(\Lambda^q(T^{*(0,1)}X)\otimes E\big)_x$ *which are polynomials in* R^{TX}, R^E (and $d\Theta$, R^L) *and their derivatives of order* $\leqslant 2r-2$ (resp. $\leqslant 2r-1, 2r$) *at* x*, such that*

$$\mathbf{b}_0 = I_{\det(\overline{W}^*)\otimes E} \tag{3.1.10}$$

and for any $k,l \in \mathbb{N}$*, there exists* $C_{k,l} > 0$ *with*

$$\Big| P_p^{0,q}(x,x) - \sum_{r=0}^{k} \mathbf{b}_r(x) p^{n-r}\Big|_{C^l(X)} \leqslant C_{k,l}\, p^{n-k-1}, \tag{3.1.11}$$

for any $p \in \mathbb{N}$*. Moreover, the expansion is uniform in that for any* $k,l \in \mathbb{N}$*, there is an integer* s *such that if all data* $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E)$ *run over a set which are bounded in* C^s *and with* g^{TX} *bounded below, there exists the constant* $C_{k,l}$ *independent of* g^{TX}*, and the* C^l*-norm in* (3.1.11) *includes also the derivative on the parameters.*

To state our main result we continue to introduce more notations. Let $\nabla^{T^{(1,0)}X}$ be the Chern connection of $(T^{(1,0)}X,\ h^{T^{(1,0)}X})$, where $h^{T^{(1,0)}X}$ is the Hermitian metric on $T^{(1,0)}X$ induced by g^{TX} in (3.1.3). We denote by $R^{T^{(1,0)}X}$ the curvature of $\nabla^{T^{(1,0)}X}$. Let $\mathbf{J} : TX \to TX$ be the almost complex structure defined by

$$\omega(U,V) = g^{TX}(\mathbf{J}U, V) \ \text{ for } U,V \in TX. \tag{3.1.12}$$

Then J commutes with $\mathbf{J}$. Let $v_1,\dots,v_n$ an orthonormal frame of $(T^{(1,0)}X, h^{T^{(1,0)}X})$ such that the subbundle W is spanned by $v_1,\dots,v_q$, and let $v^1,\dots,v^n$ be the dual frame. It is a consequence of (3.1.3) and (3.1.12) that

$$\mathbf{J}v_j = -\sqrt{-1}v_j, \text{ for } j \leqslant q; \quad \mathbf{J}v_j = \sqrt{-1}v_j, \text{ for } j \geqslant q+1. \tag{3.1.13}$$

Let $T_{\mathbf{J}}^{(1,0)}X$ and $T_{\mathbf{J}}^{(0,1)}X$ be the eigenbundles of $\mathbf{J}$ corresponding to the eignevalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Set

$$u_j = \overline{v}_j \text{ if } j \leqslant q \text{ and } u_j = v_j \text{ otherwise.} \tag{3.1.14}$$

Then $u_1, \ldots, u_n$ forms an orthonormal frame of the subbundle $T_{\mathbf{J}}^{(1,0)}X$. We denote by $u^1, \ldots, u^n$ its dual frame. Then

$$\omega = \sqrt{-1} \sum_{j=1}^{n} u^j \wedge \overline{u}^j. \tag{3.1.15}$$

Let ∇^B be the Bismut connection (see (3.2.20)) on $\Lambda(T^{*(0,1)}X)$ whose curvature is denoted by R^B. Denote by $\nabla^X \psi, \nabla^B \psi$ the covariant derivative of a tensor ψ with respect to ∇^{TX} and ∇^B, respectively. Let $e_1, \ldots, e_{2n}$ denote an orthonormal frame of (TX, g^{TX}), set

$$\left|\nabla^X \mathbf{J}\right|^2 = \sum_{i,j=1}^{2n} \left|(\nabla^X_{e_i} \mathbf{J}) e_j\right|^2, \quad \left|\nabla^B \mathbf{J}\right|^2 = \sum_{i,j=1}^{2n} \left|(\nabla^B_{e_i} \mathbf{J}) e_j\right|^2. \tag{3.1.16}$$

We denote by T_{as} the anti-symmetrization of the torsion tensor of the connection induced by the Chern connection $\nabla^{T^{(1,0)}X}$ on TX (cf. (3.2.1), (3.2.2)). Let Λ_ω be the contraction operator with the form ω. Let P be the smooth 2-form over X defined by

$$\begin{aligned} P(U,V) =& \frac{1}{2} \langle R^B(u_j, \overline{u}_j) U, V \rangle \\ &+ \frac{1}{4} (dT_{as})(u_j, \overline{u}_j, U, V) + \Big(\frac{1}{2} \mathrm{Tr}\big[R^{T^{(1,0)}X}\big] + R^E \Big)(U,V). \end{aligned} \tag{3.1.17}$$

The summation convention of summing over repeated indices is used here and throughout this paper. Note that (cf. [30, (1.2.51)]),

$$T_{as} = -\sqrt{-1}\big(\partial - \overline{\partial}\big)\Theta \quad \text{and} \quad dT_{as} = 2\sqrt{-1}\partial\overline{\partial}\Theta. \tag{3.1.18}$$

The main result of this Chapter is as follows.

Theorem 3.3. *Let X be a compact complex manifold and (L, h^L) be a holomorphic Hermitian line bundle whose curvature is non-degenerate of signature $(q, n-q)$. Let (E, h^E) be a holomorphic Hermitian vector bundle. Endow $\Omega^{0,\bullet}(X, L^p \otimes E)$ with the L^2-scalar product induced by the Riemannian metric g^{TX} defined by (3.1.3) and by h^L, h^E. Then the coefficient $\mathbf{b}_1$ from the expansion (3.1.11) of the Bergman kernel $P_p^{0,q}(\cdot,\cdot)$ on*

$(0,q)$-forms is given by

$$
\begin{aligned}
\pi\mathbf{b}_1(x) =&\Big[\frac{1}{2}R^E(u_j,\overline{u}_j)+\frac{1}{4}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](u_j,\overline{u}_j)\\
&-\frac{1}{16}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)-\frac{1}{144}\big|(\nabla^B_{u_i}\mathbf{J})u_j\big|^2\Big]I_{\det(\overline{W}^*)\otimes E} \qquad (3.1.19)\\
&+\frac{1}{72}\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}\Big\langle(\nabla^B_{u_m}\mathbf{J})u_j,u_k\Big\rangle\Big\langle(\nabla^B_{\overline{u}_m}\mathbf{J})\overline{u}_i,\overline{u}_l\Big\rangle\overline{u}^l\wedge i_{u_i}I_{\det(\overline{W}^*)\otimes E}u^j\wedge i_{\overline{u}_k}\\
&-\frac{1}{4}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big[P(\overline{u}_j,\overline{u}_k)-\frac{\sqrt{-1}}{3}\big\langle(\nabla^B\nabla^B\mathbf{J})_{(u_i,\overline{u}_i)}\overline{u}_j,\overline{u}_k\big\rangle\Big]\overline{u}^k\wedge i_{u_j}I_{\det(\overline{W}^*)\otimes E}\\
&+\frac{1}{8}\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}\Big[\frac{1}{8}(dT_{as})(\overline{u}_i,\overline{u}_j,\overline{u}_k,\overline{u}_l)-\frac{1}{15}\big\langle(\nabla^B_{u_m}\mathbf{J})\overline{u}_i,\overline{u}_l\big\rangle\cdot\big\langle(\nabla^B_{\overline{u}_m}\mathbf{J})\overline{u}_j,\overline{u}_k\big\rangle\\
&\qquad-\frac{1}{10}\big\langle(\nabla^B_{\overline{u}_m}\mathbf{J})\overline{u}_i,\overline{u}_l\big\rangle\cdot\big\langle(\nabla^B_{u_m}\mathbf{J})\overline{u}_j,\overline{u}_k\big\rangle\Big]\overline{u}^k\wedge\overline{u}^l\wedge i_{u_i}i_{u_j}I_{\det(\overline{W}^*)\otimes E}\\
&-\frac{1}{4}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big[P(u_k,u_j)-\frac{\sqrt{-1}}{3}\big\langle(\nabla^B\nabla^B\mathbf{J})_{(\overline{u}_i,u_i)}u_k,u_j\big\rangle\Big]I_{\det(\overline{W}^*)\otimes E}u^j\wedge i_{\overline{u}_k}\\
&+\frac{1}{8}\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}\Big[\frac{1}{8}(dT_{as})(u_i,u_j,u_k,u_l)-\frac{1}{15}\big\langle(\nabla^B_{\overline{u}_m}\mathbf{J})u_i,u_l\big\rangle\cdot\big\langle(\nabla^B_{u_m}\mathbf{J})u_j,u_k\big\rangle\\
&\qquad-\frac{1}{10}\big\langle(\nabla^B_{u_m}\mathbf{J})u_i,u_l\big\rangle\cdot\big\langle(\nabla^B_{\overline{u}_m}\mathbf{J})u_j,u_k\big\rangle\Big]I_{\det(\overline{W}^*)\otimes E}u^j\wedge u^i\wedge i_{\overline{u}_l}i_{\overline{u}_k}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\pi\cdot\mathrm{Tr}|_{\Lambda^q(T^{*(0,1)}X)}\big[\mathbf{b}_1(x)\big]\\
=&\frac{1}{2}R^E(u_j,\overline{u}_j)+\frac{1}{4}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](u_j,\overline{u}_j)-\frac{1}{16}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big).
\end{aligned}
\qquad (3.1.20)
$$

By integration of the expansion (3.1.11) we can compare coefficients with the Riemann-Roch-Hirzebruch formula and we can check our formula for $\mathbf{b}_1$. This will carried out in Section 3.7.

Since the explicit formula (3.1.19) seems rather lengthy, it is worthwhile to show what it reduces to in various interesting special cases.

Corollary 3.4. *If (X, g^{TX}, J) is Kähler, then we have*

$$
\begin{aligned}
\pi \mathbf{b}_1(x) = & \Big[\frac{1}{2} R^E(u_j, \overline{u}_j) + \frac{1}{4}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](u_j, \overline{u}_j) - \frac{1}{144}\big|(\nabla^X_{u_i}\mathbf{J})u_j\big|^2\Big] I_{\det(\overline{W}^*)\otimes E} \\
& + \frac{1}{72}\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n} \Big\langle (\nabla^X_{u_m}\mathbf{J})u_j, u_k\Big\rangle\Big\langle(\nabla^X_{\overline{u}_m}\mathbf{J})\overline{u}_i, \overline{u}_l\Big\rangle \overline{u}^l \wedge i_{u_i} I_{\det(\overline{W}^*)\otimes E} u^j \wedge i_{\overline{u}_k} \\
& - \frac{1}{4}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big[\Big(\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big] + R^E\Big)(\overline{u}_j, \overline{u}_k) \\
& \qquad\qquad - \frac{1}{6}\big\langle R^{TX}(u_i, \overline{u}_i)\overline{u}_j, \overline{u}_k\big\rangle\Big]\overline{u}^k \wedge i_{u_j} I_{\det(\overline{W}^*)\otimes E} \\
& - \frac{1}{4}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big[\Big(\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big] + R^E\Big)(u_k, u_j) \\
& \qquad\qquad - \frac{1}{6}\big\langle R^{TX}(u_i, \overline{u}_i)u_k, u_j\big\rangle\Big] I_{\det(\overline{W}^*)\otimes E} u^j \wedge i_{\overline{u}_k}.
\end{aligned} \tag{3.1.21}
$$

Taking the trace over $\Lambda^q(T^{*(0,1)}X)$ yields

$$
\pi \cdot \mathrm{Tr}|_{\Lambda^q(T^{*(0,1)}X)}\big[\mathbf{b}_1(x)\big] = \frac{1}{2}R^E(u_j, \overline{u}_j) + \frac{1}{4}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](u_j, \overline{u}_j). \tag{3.1.22}
$$

Corollary 3.5. *If $q = 0$, then it follows* (3.1.3) *and* (3.1.12) *that (X, g^{TX}, J) is Kähler. Then the formula* (3.1.19) *reduces to the known one* [30, (4.1.8)] *for positive line bundles:*

$$
\pi \mathbf{b}_1(x) = \frac{1}{2}R^E(v_j, \overline{v}_j) + \frac{1}{4}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](v_j, \overline{v}_j)\mathrm{Id}_E = \frac{1}{2}R^E(v_j, \overline{v}_j) + \frac{r^X}{8}\mathrm{Id}_E. \tag{3.1.23}
$$

Formula (3.1.23) follows immediately from (3.1.21).

3.2 Bergman kernel of the Hodge-Dolbeault operator

In this Section we introduce the corresponding Lichnerowicz formula for the operator D_p^2 and the Bergman kernel of the operator D_p^2. We also calculate here the curvature operator $R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$ which naturally arises in the Lichnerowicz formula.

3.2.1 Lichnerowicz formula

For any $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0}$ be the metric dual of $v_{1,0}$. Then $c(v) = \sqrt{2}(\overline{v}^*_{1,0} \wedge - i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product, respectively.

The holomorphic Hermitian connection $\nabla^{T^{(1,0)}X}$ on $T^{(1,0)}X$ induces naturally a Hermitian connection $\nabla^{T^{(0,1)}X}$ on $T^{(0,1)}X$. Set

$$
\tilde{\nabla}^{TX} = \nabla^{T^{(1,0)}X} \oplus \nabla^{T^{(0,1)}X}. \tag{3.2.1}
$$

Then $\tilde{\nabla}^{TX}$ is a Hermitian connection on $TX \otimes_{\mathbb{R}} \mathbb{C}$ and it preserves the decomposition $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$. We still denote by $\tilde{\nabla}^{TX}$ the induced connection on TX. Let T be the torsion of the connection $\tilde{\nabla}^{TX}$, and let T_{as} be the anti-symmetrization of the tensor T, i.e., for $U, V, W \in TX$,

$$T_{as}(U,V,W) = \big\langle T(U,V), W\big\rangle + \big\langle T(V,W), U\big\rangle + \big\langle T(W,U), V\big\rangle. \tag{3.2.2}$$

It is a consequence of (3.2.1) that the torsion operator T maps $T^{(1,0)}X \otimes T^{(1,0)}X$ (resp. $T^{(0,1)}X \otimes T^{(0,1)}X$) into $T^{(1,0)}X$ (resp. $T^{(0,1)}X$) and vanishes on $T^{(1,0)}X \otimes T^{(0,1)}X$.

If $\{e^1, \dots, e^{2n}\}$ denotes an orthonormal frame of T^*X, then define

$$^c(e^{i_1} \wedge \cdots \wedge e^{i_j}) = c(e_{i_1}) \cdots c(e_{i_j}), \quad \text{for } i_1 < \cdots < i_j. \tag{3.2.3}$$

In this sense ^cB is defined for any $B \in \Lambda^{i,j}$ by extending $\mathbb{C}$-linearity.

Take $U \in TX$. Let

$$\nabla_U^{B,\Lambda^{0,\bullet}} = \nabla_U^{Cl} - \frac{1}{4}\, {}^c(i_U T_{as}) \tag{3.2.4}$$

be a Hermitian connection on $\Lambda(T^{*(0,1)}X)$ induced by ∇^{Cl} and T_{as}, then $\nabla_U^{B,\Lambda^{0,\bullet}}$ preserve the $\mathbb{Z}$-grade of $\Lambda(T^{*(0,1)}X)$ (cf. [30, (1.4.27)]). If $\{v_1, \dots, v_n\}$ denotes an orthonormal frame of $T^{(1,0)}X$ as in (3.1.13), set

$$e_{2j-1} = \frac{1}{\sqrt{2}}(v_j + \overline{v}_j), \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(v_j - \overline{v}_j). \tag{3.2.5}$$

Then $\{e_1, e_2, \dots, e_{2n-1}, e_{2n}\}$ forms an orthonormal frame of TX. Set

$$\nabla^{TX} e_j = \Gamma^{TX} e_j, \quad \nabla^{\det(T^{(1,0)}X)}(v_1 \wedge \cdots \wedge v_n) = \Gamma^{\det(T^{(1,0)}X)}(v_1 \wedge \cdots \wedge v_n). \tag{3.2.6}$$

Denote by $\overline{v}^j$ the metric dual of v_j. It is a consequence of [30, (1.3.5)] that $\nabla^{B,\Lambda^{0,\bullet}}$ is given, with respect to the frame $\{\overline{v}^{j_1} \wedge \cdots \overline{v}^{j_k},\ 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$ of $\Lambda(T^{*(0,1)}X)$, by the local formula

$$d + \frac{1}{4}\big\langle \Gamma^{TX} e_i, e_j\big\rangle c(e_i)c(e_j) + \frac{1}{2}\Gamma^{\det(T^{(1,0)}X)} - \frac{1}{4}\, {}^c(i.T_{as}). \tag{3.2.7}$$

Let $\Gamma^{B,\Lambda^{0,\bullet}}$ be the connection 1-form of $\nabla^{B,\Lambda^{0,\bullet}}$ (associated to the above frame of $\Lambda(T^{*(0,1)}X)$), i.e.,

$$\Gamma^{B,\Lambda^{0,\bullet}} = \frac{1}{4}\big\langle \Gamma^{TX} e_i, e_j\big\rangle c(e_i)c(e_j) + \frac{1}{2}\Gamma^{\det(T^{(1,0)}X)} - \frac{1}{4}\, {}^c(i.T_{as}). \tag{3.2.8}$$

Denote by $\nabla^{L^p \otimes E}$ the holomorphic Hermitian connection on $L^p \otimes E$ induced by ∇^L and ∇^E. Set

$$\nabla^{B,\Lambda^{0,\bullet} \otimes L^p \otimes E} = \nabla^B \otimes 1 + 1 \otimes \nabla^{L^p \otimes E}. \tag{3.2.9}$$

Then $\nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$ is a Hermitian connection on E_p. Let $R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$ be the curvature operator of $\nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$, and let $\Delta^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$ be the Bochner Laplacian associated to $\nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$, i.e.,

$$\Delta^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E} = -\sum_{j=1}^{2n}\Big[\big(\nabla_{e_j}^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}\big)^2 - \nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_{\nabla^{TX}_{e_j}e_j}\Big]. \tag{3.2.10}$$

If $\{e_1,\dots,e_{2n}\}$ denotes an orthonormal frame of TX, then set

$$|A|^2 = \sum_{i<j<k}|A(e_i,e_j,e_k)|^2, \quad \text{for } A\in\Lambda^3(T^*X). \tag{3.2.11}$$

The following Lichnerowicz formula [30, (1.2.51) and (1.4.29)] for D_p^2 holds:

$$D_p^2 = \Delta^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E} + \frac{1}{2}pR^L(e_i,e_j)c(e_i)c(e_j) + \Phi, \tag{3.2.12}$$

where

$$\Phi = \frac{r^X}{4} + {}^c\Big(R^E + \frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big) - \frac{1}{4}\,{}^c(dT_{as}) - \frac{1}{8}\big|T_{as}\big|^2. \tag{3.2.13}$$

Definition 3.6. The Bergman kernel $P_p(x,y)(x,y\in X)$ is the smooth kernel with respect to $dv_X(y)$ of the orthogonal projection P_p from $\Omega^{0,\bullet}(X,L^p\otimes E)$ onto $\mathrm{Ker}D_p$.

Then $P_p(x,x)$ is an element of $\mathrm{End}\big(\Lambda(T^{*(0,1)}X)\otimes E\big)_x$. Moreover, it follows from (3.1.7) and (3.1.8) that $P_p^{0,q}(x,y)$ coincides with $P_p(x,y)$ for p large enough.

3.2.2 The calculation of the curvature operator $R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$

Denote by S^B the 1-form with values in the space of antisymmetric elements of $\mathrm{End}(TX)$ which satisfies for $U,V,W\in TX$,

$$\big\langle S^B(U)V,W\big\rangle = -\frac{1}{2}T_{as}(U,V,W). \tag{3.2.14}$$

Substituting (3.2.14) into (3.2.8) we get

$$\Gamma^{B,\Lambda^{0,\bullet}} = \frac{1}{4}\Big\langle\big(\Gamma^{TX}+S^B\big)e_i,e_j\Big\rangle c(e_i)c(e_j) + \frac{1}{2}\Gamma^{\det(T^{(1,0)}X)}. \tag{3.2.15}$$

Let $R^{B,\Lambda^{0,\bullet}}$ denote the curvature operator of $\nabla^{B,\Lambda^{0,\bullet}}$. For $U,V\in TX$, then

$$\begin{aligned} R^{B,\Lambda^{0,\bullet}}(U,V) =& \big(d\Gamma^{B,\Lambda^{0,\bullet}}\big)(U,V) + \big(\Gamma^{B,\Lambda^{0,\bullet}}\wedge\Gamma^{B,\Lambda^{0,\bullet}}\big)(U,V)\\ =& \big(d\Gamma^{B,\Lambda^{0,\bullet}}\big)(U,V) + \big[\Gamma^{B,\Lambda^{0,\bullet}}(U),\Gamma^{B,\Lambda^{0,\bullet}}(V)\big]. \end{aligned} \tag{3.2.16}$$

It is clear that

$$\begin{aligned}
&\frac{1}{16}\Big[\Big\langle\big(\Gamma^{TX}+S^B\big)(U)e_i,e_j\Big\rangle c(e_i)c(e_j),\Big\langle\big(\Gamma^{TX}+S^B\big)(V)e_k,e_l\Big\rangle c(e_k)c(e_l)\Big]\\
=&\frac{1}{16}\times 4\sum_{i\neq j,j\neq k}\Big\langle\big(\Gamma^{TX}+S^B\big)(U)e_i,e_j\Big\rangle\cdot\Big\langle\big(\Gamma^{TX}+S^B\big)(V)e_k,e_j\Big\rangle\Big[c(e_i)c(e_j),c(e_k)c(e_j)\Big]\\
=&\frac{1}{4}\Big\langle\big(\Gamma^{TX}+S^B\big)e_i,e_j\Big\rangle\cdot\Big\langle\big(\Gamma^{TX}+S^B\big)e_k,e_j\Big\rangle\big(c(e_i)c(e_k)-c(e_k)c(e_i)\big)\\
=&-\frac{1}{4}\Big\langle\big(\Gamma^{TX}+S^B\big)(V)\big(\Gamma^{TX}+S^B\big)e_i,e_k\Big\rangle\big(c(e_i)c(e_k)-c(e_k)c(e_i)\big)\\
=&\frac{1}{4}\Big\langle\big((\Gamma^{TX}+S^B)\wedge(\Gamma^{TX}+S^B)\big)(U,V)e_i,e_j\Big\rangle c(e_i)c(e_j).
\end{aligned}$$

Therefore,

$$\begin{aligned}
R^{B,\Lambda^{0,\bullet}}=&\frac{1}{4}\langle d\big(\Gamma^{TX}+S^B\big)e_i,e_j\rangle c(e_i)c(e_j)+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\\
&+\frac{1}{4}\Big\langle\big((\Gamma^{TX}+S^B)\wedge(\Gamma^{TX}+S^B)\big)e_i,e_j\Big\rangle c(e_i)c(e_j)\\
=&\frac{1}{4}\langle(R^{TX}+Q)e_i,e_j\rangle c(e_i)c(e_j)+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big],
\end{aligned}\tag{3.2.17}$$

where

$$Q=dS^B+S^B\wedge S^B+\big(S^B\wedge\Gamma^{TX}+\Gamma^{TX}\wedge S^B\big).\tag{3.2.18}$$

Then

$$\begin{aligned}
R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}=&R^{B,\Lambda^{0,\bullet}}+pR^L+R^E\\
=&\frac{1}{4}\langle(R^{TX}+Q)e_i,e_j\rangle c(e_i)c(e_j)+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]+pR^L+R^E.
\end{aligned}\tag{3.2.19}$$

Let us consider the Bismut connection on TX

$$\nabla^B=\nabla^{TX}+S^B.\tag{3.2.20}$$

By [5, Prop. 2.5], ∇^B preserves the metric g^{TX} and the complex structure J of TX. Then the curvature R^B is compatible with the complex structure J of TX, so is the curvature $R^{B,\Lambda^{0,\bullet}}$. Clearly,

$$R^B=R^{TX}+[\nabla^{TX},S^B]+S^B\wedge S^B.\tag{3.2.21}$$

Combining (3.2.18) and (3.2.21) yield

$$R^B=R^{TX}+Q.\tag{3.2.22}$$

Substituting (3.2.22) into (3.2.19) we obtain

$$R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}=\frac{1}{4}\langle R^Be_i,e_j\rangle c(e_i)c(e_j)+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]+pR^L+R^E.\tag{3.2.23}$$

3.2.3 Comparison between curvatures of Bismut and Levi-Civita connections

Let us first describe some properties of the tensors $\nabla^X \mathbf{J}$ and $\nabla^B \mathbf{J}$.

It is a consequence of (3.1.12) that $\mathbf{J}, \nabla^X_\bullet \mathbf{J}, (\nabla^X\nabla^X\mathbf{J})_{(\bullet,\bullet)}$ are skew-adjoint endomorphisms of TX. Also by (3.1.12), we find that for $U, V, W \in TX$,

$$\begin{aligned} \Big\langle (\nabla^X_U \mathbf{J})V, W\Big\rangle =& (\nabla^X_U w)(V,W), \\ \Big\langle (\nabla^X_U \mathbf{J})V, W\Big\rangle + \Big\langle (\nabla^X_V \mathbf{J})W, U\Big\rangle +& \Big\langle (\nabla^X_W \mathbf{J})U, V\Big\rangle = dw(U,V,W) = 0. \end{aligned} \tag{3.2.24}$$

By the definition of $(\nabla^X\nabla^X\mathbf{J})_{(U,V)}$,

$$\begin{aligned} (\nabla^X\nabla^X\mathbf{J})_{(U,V)} - (\nabla^X\nabla^X\mathbf{J})_{(V,U)} &= \big[R^{TX}(U,V), \mathbf{J}\big], \\ \mathbf{J}\cdot(\nabla^X\nabla^X\mathbf{J})_{(U,V)} + (\nabla^X_U\mathbf{J})\cdot(\nabla^X_V\mathbf{J}) &+ (\nabla^X_V\mathbf{J})\cdot(\nabla^X_U\mathbf{J}) \\ &+ (\nabla^X\nabla^X\mathbf{J})_{(U,V)}\cdot\mathbf{J} = 0. \end{aligned} \tag{3.2.25}$$

From (3.2.24), we have for $Y \in TX$,

$$\Big\langle (\nabla^X\nabla^X\mathbf{J})_{(Y,U)}V, W\Big\rangle + \Big\langle (\nabla^X\nabla^X\mathbf{J})_{(Y,V)}W, U\Big\rangle + \Big\langle (\nabla^X\nabla^X\mathbf{J})_{(Y,W)}U, V\Big\rangle = 0. \tag{3.2.26}$$

For the other tensor $\nabla^B\mathbf{J}$, we have for $U, V, W \in TX$,

$$\Big\langle (\nabla^B_U \mathbf{J})V, W\Big\rangle = \big(\nabla^B_U \omega\big)(V,W) \tag{3.2.27}$$

and

$$\begin{aligned} (\nabla^B\nabla^B\mathbf{J})_{(U,V)} - (\nabla^B\nabla^B\mathbf{J})_{(V,U)} &= \big[R^{B}(U,V), \mathbf{J}\big], \\ \mathbf{J}\cdot(\nabla^B\nabla^B\mathbf{J})_{(U,V)} + (\nabla^B_U\mathbf{J})\cdot(\nabla^B_V\mathbf{J}) &+ (\nabla^B_V\mathbf{J})\cdot(\nabla^B_U\mathbf{J}) \\ &+ (\nabla^B\nabla^B\mathbf{J})_{(U,V)}\cdot\mathbf{J} = 0. \end{aligned} \tag{3.2.28}$$

Since the torsion ∇^B is not torsion-free, then the analogue of the second equality in (3.2.24) for the tensor $\nabla^B\mathbf{J}$ does not hold, neither does the analogue of (3.2.26). Let $T^{*(1,0)}_{\mathbf{J}}X$ and $T^{*(0,1)}_{\mathbf{J}}X$ be the dual bundle of $T^{(1,0)}_{\mathbf{J}}X$ and $T^{(0,1)}_{\mathbf{J}}X$, respectively. By the definition of $\nabla^X_\bullet\mathbf{J}$ and (3.2.24),

$$\big\langle (\nabla^X_\bullet\mathbf{J})\cdot,\cdot\big\rangle \text{ is of type } \big(T^{*(1,0)}_{\mathbf{J}}X\big)^{\otimes 3} \oplus \big(T^{*(0,1)}_{\mathbf{J}}X\big)^{\otimes 3}. \tag{3.2.29}$$

On the other hand, the tensor $\nabla^B\mathbf{J}$ satisfies the following properties.

Proposition 3.7. $\nabla^B\mathbf{J}$ *preserves* $T^{(1,0)}X$ *and* $T^{(0,1)}X$. *Furthermore, it exchanges the subbundle* W *and* $W^\perp$.

Proof. It is a consequence of the facts that $\nabla^B J = 0$ and $J\mathbf{J} = \mathbf{J}J$ that

$$J(\nabla^B \mathbf{J}) = (\nabla^B \mathbf{J})J, \tag{3.2.30}$$

which implies immediately that $\nabla^B \mathbf{J}$ preserves $T^{(1,0)}X$ and $T^{(0,1)}X$. Clearly by (3.1.13) for $U \in TX$ and $1 \leqslant j, k \leqslant q$,

$$\Big\langle (\nabla^B_U \mathbf{J})v_j, \overline{v}_k \Big\rangle = \Big\langle \nabla^B_U(\mathbf{J}v_j), \overline{v}_k \Big\rangle + \Big\langle \nabla^B_U v_j, \mathbf{J}\overline{v}_k \Big\rangle = 0. \tag{3.2.31}$$

This completes the proof of Proposition 3.7. □

Lemma 3.8.

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big|\big\langle (\nabla^B_{\overline{u}_i}\mathbf{J})u_j, u_k\big\rangle\Big|^2 &= 2\sum_{i,j,k=1}^{n} \Big|\big\langle S^B(\overline{u}_i)u_j, u_k\big\rangle\Big|^2, \\ \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big|\big\langle (\nabla^B_{u_i}\mathbf{J})u_j, u_k\big\rangle\Big|^2 &= \frac{1}{4}\big|\nabla^B \mathbf{J}\big|^2 - 2\sum_{i,j,k=1}^{n} \Big|\big\langle S^B(\overline{u}_i)u_j, u_k\big\rangle\Big|^2. \end{aligned} \tag{3.2.32}$$

Proof. In view of Proposition 3.7 and (3.2.29), we find

$$\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big|\big\langle (\nabla^B_{\overline{u}_i}\mathbf{J})u_j, u_k\big\rangle\Big|^2 = \frac{1}{2}\sum_{i,j,k=1}^{n} \Big|\big\langle (\nabla^B_{\overline{u}_i}\mathbf{J})u_j, u_k\big\rangle\Big|^2 = 2\sum_{i,j,k=1}^{n} \Big|\big\langle S^B(\overline{u}_i)u_j, u_k\big\rangle\Big|^2. \tag{3.2.33}$$

Again by Proposition 3.7 we get

$$\begin{aligned} \big|\nabla^B \mathbf{J}\big|^2 &= 2\sum_{i,j=1}^{n} \big|(\nabla^B_{\overline{u}_i}\mathbf{J})u_j\big|^2 + 2\sum_{i,j=1}^{n} \big|(\nabla^B_{u_i}\mathbf{J})u_j\big|^2 \\ &= 4\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big|\big\langle (\nabla^B_{\overline{u}_i}\mathbf{J})u_j, u_k\big\rangle\Big|^2 + 4\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n} \Big|\big\langle (\nabla^B_{u_i}\mathbf{J})u_j, u_k\big\rangle\Big|^2. \end{aligned} \tag{3.2.34}$$

Combining (3.2.33) and (3.2.34), we obtain the second equality of (3.2.32). This completes the proof of Lemma 3.8. □

The main result in this subsection is the following difference formula (3.2.35) between the two curvatures R^B and R^{TX}.

Proposition 3.9.

$$\begin{aligned} &\Big\langle \big(R^B - R^{TX}\big)(u_i, \overline{u}_i)u_j, \overline{u}_j \Big\rangle \\ =&\frac{1}{8}\Big(\big|\nabla^B \mathbf{J}\big|^2 - \big|\nabla^X \mathbf{J}\big|^2\Big) + \frac{1}{4}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big) - 2\sum_{i,j,k=1}^{n} \Big|\big\langle S^B(\overline{u}_i)u_j, u_k\big\rangle\Big|^2. \end{aligned} \tag{3.2.35}$$

To prove Proposition 3.9 we need the following three Lemmas.

Lemma 3.10.

$$\begin{aligned}&\Big\langle\big(R^B-R^{TX}\big)(u_i,\overline{u}_i)u_j,\overline{u}_j\Big\rangle\\&=\sum_{i,j,k=1}^n\Big|\big\langle S^B(u_i)u_j,u_k\big\rangle\Big|^2-\sum_{i,j,k=1}^n\Big|\big\langle S^B(\overline{u}_i)u_j,u_k\big\rangle\Big|^2+\frac{1}{16}\Lambda_\omega\Lambda_\omega(dT_{as}).\end{aligned}\tag{3.2.36}$$

Proof. One verifies directly that

$$\begin{aligned}&\Big\langle\big(R^B-R^{TX}\big)(u_i,\overline{u}_i)u_j,\overline{u}_j\Big\rangle\\=&\Big\langle S^B(u_i)u_j,S^B(\overline{u}_i)\overline{u}_j\Big\rangle-\Big\langle S^B(\overline{u}_i)u_j,S^B(u_i)\overline{u}_j\Big\rangle\\&+\Big\langle\big(\nabla^X_{u_i}S^B\big)(\overline{u}_i)u_j,\overline{u}_j\Big\rangle-\Big\langle\big(\nabla^X_{\overline{u}_i}S^B\big)(u_i)u_j,\overline{u}_j\Big\rangle.\end{aligned}\tag{3.2.37}$$

By (3.2.14), we obtain

$$\begin{aligned}\Big\langle\big(\nabla^X_{u_i}S^B\big)(\overline{u}_i)u_j,\overline{u}_j\Big\rangle&=-\frac{1}{2}\big(\nabla^X_{u_i}T_{as}\big)(\overline{u}_i,u_j,\overline{u}_j),\\\Big\langle\big(\nabla^X_{\overline{u}_i}S^B\big)(u_i)u_j,\overline{u}_j\Big\rangle&=-\frac{1}{2}\big(\nabla^X_{\overline{u}_i}T_{as}\big)(u_i,u_j,\overline{u}_j).\end{aligned}\tag{3.2.38}$$

Substituting (3.2.38) into (3.2.37) yields (3.2.36). □

Lemma 3.11.

$$\begin{aligned}&\frac{1}{8}\Big(\big|\nabla^B\mathbf{J}\big|^2-\big|\nabla^X\mathbf{J}\big|^2\Big)\\&=\sum_{i,j,k=1}^n\Big|\big\langle S^B(u_i)u_j,u_k\big\rangle\Big|^2+\sum_{i,j,k=1}^n\Big|\big\langle S^B(\overline{u}_i)u_j,u_k\big\rangle\Big|^2\\&\quad+\frac{\sqrt{-1}}{2}\Big\langle S^B(u_i)u_j,(\nabla^X_{\overline{u}_i}\mathbf{J})\overline{u}_j\Big\rangle-\frac{\sqrt{-1}}{2}\Big\langle S^B(\overline{u}_i)\overline{u}_j,(\nabla^X_{u_i}\mathbf{J})u_j\Big\rangle.\end{aligned}\tag{3.2.39}$$

Proof. By (3.2.29) we obtain

$$\big|\nabla^X\mathbf{J}\big|^2=2\sum_{i,j=1}^n\big|(\nabla^X_{u_i}\mathbf{J})u_j\big|^2,\quad\big|\nabla^B\mathbf{J}\big|^2=2\sum_{i,j=1}^n\big|(\nabla^B_{u_i}\mathbf{J})u_j\big|^2+2\sum_{i,j=1}^n\big|(\nabla^B_{\overline{u}_i}\mathbf{J})u_j\big|^2.\tag{3.2.40}$$

A direct computation leads to

$$\begin{aligned}&\big|\nabla^B\mathbf{J}\big|^2-\big|\nabla^X\mathbf{J}\big|^2\\=&2\sum_{i,j=1}^n\Big|[S^B(u_i),\mathbf{J}]u_j\Big|^2+2\sum_{i,j=1}^n\Big|[S^B(\overline{u}_i),\mathbf{J}]u_j\Big|^2\\&+2\Big\langle[S^B(u_i),\mathbf{J}]u_j,\big(\nabla^X_{\overline{u}_i}\mathbf{J}\big)\overline{u}_j\Big\rangle+2\Big\langle[S^B(\overline{u}_i),\mathbf{J}]\overline{u}_j,\big(\nabla^X_{u_i}\mathbf{J}\big)u_j\Big\rangle.\end{aligned}\tag{3.2.41}$$

Clearly,

$$\sum_{i,j=1}^{n}\Big|[S^B(u_i),\mathbf{J}]u_j\Big|^2=\sum_{i,j,k=1}^{n}\Big|\big\langle[S^B(u_i),\mathbf{J}]u_j,u_k\big\rangle\Big|^2=4\sum_{i,j,k=1}^{n}\Big|\big\langle S^B(u_i)u_j,u_k\big\rangle\Big|^2, \quad (3.2.42)$$

and

$$\sum_{i,j=1}^{n}\Big|[S^B(\overline{u}_i),\mathbf{J}]u_j\Big|^2=\sum_{i,j,k=1}^{n}\Big|\big\langle[S^B(\overline{u}_i),\mathbf{J}]u_j,u_k\big\rangle\Big|^2=4\sum_{i,j,k=1}^{n}\Big|\big\langle S^B(\overline{u}_i)u_j,u_k\big\rangle\Big|^2. \quad (3.2.43)$$

Substituting (3.2.42) and (3.2.43) into (3.2.41) yields (3.2.39). □

Lemma 3.12.

$$\begin{aligned}\frac{1}{4}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)=&\frac{1}{16}\Lambda_\omega\Lambda_\omega(dT_{as})-\big\langle S^B(u_i)u_j,u_k\big\rangle\big\langle\nabla^{TX}_{\overline{u}_i}\overline{u}_j,\overline{u}_k\big\rangle\\&-\big\langle S^B(\overline{u}_i)\overline{u}_j,\overline{u}_k\big\rangle\big\langle\nabla^{TX}_{u_i}u_j,u_k\big\rangle. \quad (3.2.44)\end{aligned}$$

Proof. Clearly

$$\begin{aligned}-\frac{1}{4}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)=&\frac{\sqrt{-1}}{4}d(\Lambda_\omega T_{as})(u_i,\overline{u}_i) \quad (3.2.45)\\=&\frac{1}{2}\nabla_{u_i}\big(T_{as}(u_j,\overline{u}_j,\overline{u}_i)\big)-\frac{1}{2}\nabla_{\overline{u}_i}\big(T_{as}(u_j,\overline{u}_j,u_i)\big)-\frac{1}{2}T_{as}(u_j,\overline{u}_j,[u_i,\overline{u}_i]).\end{aligned}$$

By (3.2.14) we obtain

$$\begin{aligned}-\frac{1}{4}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)=&-\frac{1}{16}\Lambda_\omega\Lambda_\omega(dT_{as})+\big\langle S^B(\overline{u}_i)\overline{u}_j,\nabla^{TX}_{u_i}u_j\big\rangle+\big\langle S^B(u_j)\overline{u}_i,\nabla^{TX}_{u_i}\overline{u}_j\big\rangle\\&-\big\langle S^B(u_i)\overline{u}_j,\nabla^{TX}_{\overline{u}_i}u_j\big\rangle+\big\langle S^B(u_i)u_j,\nabla^{TX}_{\overline{u}_i}\overline{u}_j\big\rangle. \quad (3.2.46)\end{aligned}$$

Clearly,

$$\begin{aligned}\big\langle S^B(\overline{u}_i)\overline{u}_j,\nabla^{TX}_{u_i}u_j\big\rangle+\big\langle S^B(u_j)\overline{u}_i,\nabla^{TX}_{u_i}\overline{u}_j\big\rangle=&\big\langle S^B(\overline{u}_i)\overline{u}_j,\overline{u}_k\big\rangle\cdot\big\langle\nabla^{TX}_{u_i}u_j,u_k\big\rangle, \quad (3.2.47)\\\big\langle S^B(u_i)u_j,\nabla^{TX}_{\overline{u}_i}\overline{u}_j\big\rangle-\big\langle S^B(u_i)\overline{u}_j,\nabla^{TX}_{\overline{u}_i}u_j\big\rangle=&\big\langle S^B(u_i)u_j,u_k\big\rangle\cdot\big\langle\nabla^{TX}_{\overline{u}_i}\overline{u}_j,\overline{u}_k\big\rangle.\end{aligned}$$

Substituting (3.2.47) into (3.2.46) yields (3.2.44). The proof of Lemma 3.12 is complete. □

Proof of Proposition 3.9. Formula (3.2.35) follows immediately from (3.2.36), (3.2.39) and (3.2.44). □

3.3 The spectral gap of D_p^2

In this Section, we establish the spectral gap property of the Hodge-Dolbeault operator D_p. This property serves as an essential ingredient for the existence of the asymptotic expansion of the Bergman kernel.

If A is a 2-form, then

$$\begin{aligned}\frac{1}{4}A(e_i,e_j)c(e_i)c(e_j) =& -\frac{1}{2}A(v_j,\overline{v}_j) + A\big(v_j,\overline{v}_k\big)\overline{v}^k \wedge i_{\overline{v}_j} \\ &+ \frac{1}{2}A(v_j,v_k)i_{\overline{v}_j}i_{\overline{v}_k} + \frac{1}{2}A(\overline{v}_j,\overline{v}_k)\overline{v}^j \wedge \overline{v}^k \wedge .\end{aligned} \tag{3.3.1}$$

Set

$$\begin{aligned}\omega_{d,x} =& -2\pi\sum_{j=1}^{q} i_{\overline{v}_j} \wedge \overline{v}^j - 2\pi \sum_{j=q+1}^{n} \overline{v}^j \wedge i_{\overline{v}_j}, \\ \tau_x =& \pi \mathrm{Tr}\big|_{TX}(-\mathbf{J}^2)^{1/2} = 2n\pi.\end{aligned} \tag{3.3.2}$$

From (3.1.12), (3.1.13) and (3.3.1), we have

$$\frac{1}{2}R^L(e_i,e_j)c(e_i)c(e_j) = -2\omega_d - \tau. \tag{3.3.3}$$

Set

$$\Omega^{0,\neq q}(X, L^p\otimes E) = \sum_{j\neq q}\Omega^{0,j}(X, L^p\otimes E). \tag{3.3.4}$$

The following result is due to Ma and Marinescu, see [29, Theorem 1.1].

Theorem 3.13. *There exists $C_L > 0$ such that for any $p \in \mathbb{N}$,*

$$\big\|D_p^2 s\big\|_{L^2}^2 \geqslant (4p\pi - C_L)\big\|s\big\|_{L^2}^2, \quad \forall\ s \in \Omega^{0,\neq q}(X, L^p\otimes E). \tag{3.3.5}$$

Proof. We give the proof for the reader's convenience. Substituting (3.3.3) into (3.2.12) we get,

$$\big\|D_p^2 s\big\|_{L^2}^2 = \big\|\nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}s\big\|_{L^2}^2 - p\big\langle \tau s, s\big\rangle - 2p\big\langle \omega_d s, s\big\rangle + \big\langle \Phi s, s\big\rangle. \tag{3.3.6}$$

We first claim that there exists $C_1 > 0$ such that for every $s \in \Omega^{0,\bullet}(X, L^p\otimes E)$,

$$\big\|\nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}s\big\|_{L^2}^2 - p\big\langle \tau s, s\big\rangle \geqslant -C_1\big\|s\big\|_{L^2}^2. \tag{3.3.7}$$

In fact, let us now consider the almost complex manifold $(X,\mathbf{J})$. Recall that

$$u_j = \overline{v}_j,\ \text{for } j \leqslant q \text{ and } u_j = v_j,\ \text{for } j \geqslant q. \tag{3.3.8}$$

Then $\{u_1,\ldots,u_n\}$ spans the eigenbundles of $\mathbf{J}$ corresponding to the eigenvalue $\sqrt{-1}$ and the positive condition [30, (1.5.21)] holds for R^L, i.e.,

$$R^L(u_j,\overline{u}_j) = 2\pi,\ \text{and } R^L(u_j,u_k) = 0 = R^L(\overline{u}_j,\overline{u}_k). \tag{3.3.9}$$

Let $(E', h^{E'})$ be a Hermitian vector bundle on $(X,\mathbf{J})$, and let $\nabla^{E'}$ denote a Hermitian connection on $(E', h^{E'})$. Denote by $\nabla^{L^p\otimes E'}$ the Hermitian connection on $L^p\times E'$ induced by ∇^L and $\nabla^{E'}$. It is a consequence of (3.3.9) that [30, (1.5.30)] holds, i.e., there exists $C_2 > 0$ such that for every $s \in C^\infty(X, L^p \times E')$,

$$\left\|\nabla^{L^p\otimes E'} s\right\|_{L^2}^2 - p\langle \tau' s, s\rangle \geqslant -C_2\left\|s\right\|_{L^2}^2, \tag{3.3.10}$$

where

$$\tau' = \sum_{j=1}^n R^L(u_j, \overline{u}_j) = 2n\pi = \tau. \tag{3.3.11}$$

Then (3.3.7) follows from (3.3.10) by taking $E' = \Lambda(T^{*(0,1)}X) \otimes E$ and $\nabla^{L^p\otimes E'} = \nabla^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$.

In view of (3.3.2), for every $s \in \Omega^{0,\neq q}(X, L^p \otimes E)$,

$$\langle \omega_d s, s\rangle \leqslant -2\pi\left\|s\right\|_{L^2}^2. \tag{3.3.12}$$

Clearly there exists $C_3 > 0$ such that for every $s \in \Omega^{0,\bullet}(X, L^p \otimes E)$,

$$\langle \Phi s, s\rangle \geqslant -C_3\left\|s\right\|_{L^2}^2. \tag{3.3.13}$$

Substituting (3.3.7), (3.3.12) and (3.3.13) into (3.3.6), we get (3.3.5). The proof of Theorem 3.13 is complete. □

The following spectral gap, i.e., (3.3.14) (cf. [29, Theorem 1.2]) plays an essential role in the asymptotic expansion of the Bergman kernel of the Hodge-Dolbeault operators. For any operator P, we denote by Spec(P) the spectrum of P. It is a consequence of Theorem 3.13 that

Corollary 3.14.

$$\mathrm{Spec}(D_p^2) \subset \{0\} \cup [4\pi p - C_L, +\infty). \tag{3.3.14}$$

Then, (3.1.8) *follows immediately from* (3.3.14) *for p large enough.*

3.4 Diagonal asymptotic expansion of Bergman Kernel

This Section is devoted to the study of the near diagonal asymptotic expansion of the Bergman kernel. It is written along the lines of [30, Chapter 4]. In subsection 3.4.1, we explain that our problem is local. The localization of the problem allows us to extend the Hodge-Dolbeault operator from a small neighborhood of 0 to the whole space of $\mathbb{R}^{2n}$, such that the spectral gap property still holds for the operator $(D_p^{c,A_0})^2$, where D_p^{c,A_0} denotes the extension operator on $\mathbb{R}^{2n}$ of the Hodge-Dolbeault operator D_p. This is done in subsection 3.4.2. We also derive there the Taylor expansion of the rescaled operator $\mathcal{L}_2^t$ of $(D_p^{c,A_0})^2$. Subsection 3.4.3 is devoted to the Sobolev estimate on the resolvent $(\lambda - \mathcal{L}_2^t)^{-1}$. In subsection 3.4.4, we derive the uniform estimate on the Bergman kernel of the rescaled operator $\mathcal{L}_2^t$ and then establish its near diagonal estimates in Theorem 3.30. In subsection 3.4.5, we examine the Bergman kernel of the limit operator $\mathcal{L}_2^0$ of the rescaled operator $\mathcal{L}_2^t$. Then we finish our proof of Theorem 3.2 in subsection 3.4.6. Moreover, an explicit formula (3.4.224) is given there for the second coefficient of the asymptotic expansion in Theorem 3.2.

3.4.1 Localization of the problem

For $x \in X, \varepsilon > 0$, denote by $B^X(x,\varepsilon)$ the open ball in X with the center x and radius ε. Choose ε small enough such that a normal coordinate around $B^X(x,\varepsilon)$ is available. Since X is compact, then there exists $\{x_1,\dots,x_{N_0}\}$ such that the union of $B^X(x_i,\varepsilon)$ forms a covering of X. Set $U_i = B^X(x_i,\varepsilon)$. On U_i, we identify $L_Z, E_Z, \Lambda(T_Z^{*(0,1)}X)$ to $L_{x_i}, E_{x_i}, \Lambda(T_{x_i}^{*(0,1)}X)$ by parallel transport with respect to the connections ∇^L, ∇^E, $\nabla^{B,\Lambda^{0,\bullet}}$, respectively, along the curve $t \mapsto tZ, t\in[0,1]$. This induces a trivialization of E_p on U_{x_i}.

Let $\{e_1,\dots,e_{2n}\}$ be an orthonormal basis of $T_{x_i}X$. Denote by ∇_U the ordinary differential operator on $T_{x_i}X$ in the direction U. Let $\{\psi_1,\dots,\psi_{N_0}\}$ be a partition of unity subordinate to $\{U_1,\dots,U_{N_0}\}$. For $l\in\mathbb{N}$, we define a Sobolev norm on the l-th Sobolev space $\mathbf{H}^l(X,E_p)$ by

$$\|s\|_l = \sum_{i=1}^{N_0}\sum_{k=0}^{l}\sum_{i_1,\cdots,i_k=1}^{2n} \|\nabla_{e_{i_1}}\cdots\nabla_{e_{i_k}}(\psi_i s)\|_{L^2}^2. \tag{3.4.1}$$

It is clear that the norm $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{L^2}$ as in (3.1.5).

Lemma 3.15. *For any $m\in\mathbb{N}$, there exists $C_m>0$ such that for any $s\in\mathbf{H}^{2m+2}(X,E_p)$,*

$$\|s\|_{2m+2} \leqslant C_m p^{4m+4}\sum_{j=0}^{m+1} p^{-2j}\|D_p^{2j}s\|_{L^2}. \tag{3.4.2}$$

Proof. Let $\Gamma^L,\Gamma^E,\Gamma^{B,\Lambda^{0,\bullet}}$ be the connection form of ∇^L,∇^E and $\nabla^{B,\Lambda^{0,\bullet}}$, respectively, with respect to any fixed frame for L,E and $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve $t\mapsto tZ, t\in[0,1]$. From [30, (1.4.17) and (1.4.28)], we have on U_i,

$$D_p = \sum_{j=1}^{2n} c(\tilde{e}_j)\big(\nabla_{\tilde{e}_j} + p\Gamma^L(\tilde{e}_j) + \Gamma^B(\tilde{e}_j) + \Gamma^{B,\Lambda^{0,\bullet}}(\tilde{e}_j)\big) + \frac{1}{2}\,{}^c(T_{as}), \tag{3.4.3}$$

where $\tilde{e}_j$ denotes the parallel transport of e_j with respect to ∇^{TX} along the curve $t\mapsto tZ, t\in[0,1]$. Then

$$D_p^2 = -(\nabla_{\tilde{e}_j})^2 + pc(\tilde{e}_j)c(\tilde{e}_k)\Gamma^L(\tilde{e}_j)\nabla_{\tilde{e}_k} - p^2\big(\Gamma^L(\tilde{e}_j)\big)^2 + I_1, \tag{3.4.4}$$

where I_1 is a differential operator of order 1 with values in $\mathrm{End}\big(\Lambda(T_{x_i}^{*(0,1)}X)\otimes E_{x_i}\big)$ and I_1 is independent of p. From (3.4.1) and (3.4.4),

$$\begin{aligned}\|s\|_2^2 \leqslant& C\Big(\|\psi_i s\|_{L^2}^2 + \|\sum_{j=1}^{2n}(\nabla_{\tilde{e}_j})^2(\psi_i s)\|_{L_2}^2\Big)\\ \leqslant& C\Big(\|D_p^2 s\|_{L^2}^2 + p^4\|s\|_{L^2}^2\Big).\end{aligned} \tag{3.4.5}$$

That is

$$\|s\|_2 \leqslant C\Big(\|D_p^2 s\|_{L^2} + p^2\|s\|_{L^2}\Big). \tag{3.4.6}$$

Let Q be a differential operator of order m with scalar principal symbol and with compact support in U_i. Then

$$\big[D_p, Q\big] = p\big[c(\tilde{e}_j)\Gamma^L(\tilde{e}_j), Q\big] + \big[c(\tilde{e}_j)\big(\Gamma^L(\tilde{e}_j) + \Gamma^{B,\Lambda^{0,\bullet}}(\tilde{e}_j) + \Gamma^E(\tilde{e}_j)\big) + \frac{1}{2}\,{}^c(T_{as}), Q\big], \tag{3.4.7}$$

where the two summand on the righthand side of the equation are differential operators of order $m-1, m$, respectively. Clearly,

$$\big[D_p^2, Q\big] = \big[D_p, Q\big]D_p + D_p\big[D_p^2, Q\big]. \tag{3.4.8}$$

Combining (3.4.6), (3.4.7) and (3.4.8), we have

$$\begin{aligned}\|Qs\|_2 \leqslant & C\Big(\|D_p^2 Qs\|_{L^2} + p^2\|Qs\|_{L^2}\Big) \\ \leqslant & C\Big(\|QD_p^2 s\|_{L^2} + p^2\|Qs\|_{L^2} + \|s\|_{2m+1}\Big).\end{aligned} \tag{3.4.9}$$

By (3.4.9) there exists $C_m > 0$ such that

$$\|s\|_{2m+2} \leqslant C_m\Big(\|D_p^2 s\|_{2m} + p^2\|s\|_{2m+1}\Big). \tag{3.4.10}$$

Similarly,

$$\|s\|_{2m+1} \leqslant C_m\Big(\|D_p^2 s\|_{2m-1} + p^2\|s\|_{2m}\Big). \tag{3.4.11}$$

Substituting (3.4.11) into (3.4.10) we get

$$\begin{aligned}\|s\|_{2m+2} \leqslant & C_m\Big(\|D_p^2 s\|_{2m} + p^2\|D_p^2 s\|_{2m-1} + p^4\|s\|_{2m}\Big) \\ \leqslant & C_m p^2\Big(\|D_p^2 s\|_{2m} + p^2\|s\|_{2m}\Big).\end{aligned} \tag{3.4.12}$$

Replacing m by $m-1$ in (3.4.12),

$$\|s\|_{2m} \leqslant C_m p^2\Big(\|D_p^2 s\|_{2m-2} + p^2\|s\|_{2m-2}\Big). \tag{3.4.13}$$

Then

$$\|D_p^2 s\|_{2m} \leqslant C_m p^2\Big(\|D_p^4 s\|_{2m-2} + p^2\|D_p^2 s\|_{2m-2}\Big). \tag{3.4.14}$$

Substituting (3.4.13) and (3.4.14) into (3.4.12) we find

$$\|s\|_{2m+2} \leqslant C_m p^4\Big(\|D_p^4 s\|_{2m-2} + p^2\|D_p^2 s\|_{2m-2} + p^4\|s\|_{2m-2}\Big). \tag{3.4.15}$$

Similarly,

$$\big\|s\big\|_{2m+2} \leqslant C_m p^6\Big(\big\|D_p^6 s\big\|_{2m-4} + p^2\big\|D_p^4 s\big\|_{2m-4} + p^4\big\|D_p^2 s\big\|_{2m-4} + p^6\big\|s\big\|_{2m-4}\Big). \quad (3.4.16)$$

We finally get

$$\begin{aligned}\big\|s\big\|_{2m+2} \leqslant & C_m p^{2m+2}\Big(\big\|D_p^{2m+2}s\big\|_0 + p^2\big\|D_p^{2m}s\big\|_0 + \cdots + p^{2m+2}\big\|s\big\|_0\Big)\\ =& C_m p^{4m+4}\sum_{j=0}^{m+1} p^{-2j}\big\|D_p^{2j}s\big\|_0.\end{aligned} \quad (3.4.17)$$

The proof of Lemma 3.15 is complete. □

Let $\theta : \mathbb{R} \to [0,1]$ be a smooth even function given by

$$\theta(v) = 1, \text{ if } \big|v\big| \leqslant \frac{\varepsilon}{2} \text{ and } \theta(v) = 0, \text{ if } \big|v\big| > \varepsilon. \quad (3.4.18)$$

Set

$$F(a) = \Big(\int_{-\infty}^{\infty}\theta(v)dv\Big)^{-1}\int_{-\infty}^{\infty} e^{iva}\theta(v)dv. \quad (3.4.19)$$

Then $F(0) = 1$ and $F(a)$ lies in Schwartz space $\mathcal{S}(\mathbb{R})$, i.e., for any multi-index α, β there exists $C_{\alpha,\beta} > 0$ such that

$$\sup_{a\in\mathbb{R}}\Big|a^\alpha \frac{\partial^\beta}{\partial a^\beta}F(a)\Big| \leqslant C_{\alpha,\beta}. \quad (3.4.20)$$

Proposition 3.16. *For any $l, m \in \mathbb{N}$, there exist $C_{l,m} > 0$ such that for $x, y \in X$,*

$$\begin{aligned}&\big|F(D_p)(x,y) - P_p(x,y)\big|_{C^m} \leqslant C_{l,m}p^{-l},\\ &\big|P_p(x,y)\big|_{C^m} \leqslant C_{l,m}p^{-l}, \text{ if } d(x,y) \geqslant \varepsilon,\end{aligned} \quad (3.4.21)$$

where $\big|\cdot\big|_{C^m}$ is induced by ∇^L, ∇^E and h^L, h^E, g^{TX}.

Proof. For $a \in \mathbb{R}$, set

$$\phi_p(a) = \chi_{[\sqrt{2\pi p},\infty)}(|a|)F(a), \quad (3.4.22)$$

where $\chi_{[\sqrt{2\pi p},\infty)}(\cdot)$ is the characteristic function of the subset $[\sqrt{2\pi p},\infty)$ of $\mathbb{R}$.

It is a consequence of (3.3.14) that if $p > \frac{C_L}{2\pi}$, then

$$\operatorname{Spec}(D_p^2) \subset \big\{0\big\} \cup [2\pi p, \infty) \quad (3.4.23)$$

and

$$F(D_p) - P_p = \phi_p(D_p). \quad (3.4.24)$$

Set $\beta = 0$ in (3.4.20) we get

$$\sup_{a\in\mathbb{R}} \left|a\right|^m \left|F(a)\right| \leqslant C_m. \tag{3.4.25}$$

It is known that $\mathrm{Spec}(D_p^2)$ is discrete and if we denote by $\{\lambda_1, \dots, \lambda_k, \dots\}$ the set of the eigenvalues D_p^2, then $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$, $\lambda_k \to +\infty$ as $k \to \infty$ and the λ_k-eigenspace E_{λ_k} is of finite dimension. Let $\{\sigma_k^j\}$ denote an orthonormal basis of E_{λ_k}, then $D_p\sigma_k^j = \pm\sqrt{\lambda_k}\sigma_k^j$. Every $s \in \Omega^{0,\bullet}(X, L^p \otimes E)$ can be written

$$s = \sum_{k,j} \langle s, \sigma_k^j\rangle \sigma_k^j. \tag{3.4.26}$$

Then for any $m_1, m_2, l \in \mathbb{N}$,

$$\begin{aligned}\left\|D_p^{m_1}\phi_p(D_p)D_p^{m_2}s\right\|_{L^2} =& \left\|D_p^{m_1+m_2}\phi_p(D_p)s\right\|_{L^2}\\ =&\Big\|\sum_{k,j}\lambda_k^{\frac{m_1+m_2}{2}}\phi_p(\sqrt{\lambda_k})\langle s, \sigma_k^j\rangle\sigma_k^j\Big\|_{L^2}\\ \leqslant& \sup_{\lambda_k}\Big|\lambda_k^{\frac{m_1+m_2}{2}}\phi_p(\sqrt{\lambda_k})\Big| \times \Big\|\sum_{k,j}\langle s, \sigma_k^j\rangle\sigma_k^j\Big\|_{L^2}\\ \leqslant& C_{l,m_1,m_2}p^{-l}\left\|s\right\|_{L^2}. \end{aligned} \tag{3.4.27}$$

Let Q be a differential operator of order m_1 with scalar principle symbol and with compact support in U_i, then its adjoint Q^* is also a differential operator of order m_1 with scalar principle symbol and with compact support in U_i. From (3.4.2) and (3.4.27), we have

$$\begin{aligned}\left\|Q^*\phi_p(D_p)D_p^{m_2}s'\right\|_{L^2} \leqslant& C_{m_1}p^{2m_1+2}\sum_{j=0}^{[\frac{m_1}{2}]+1} p^{-2j}\left\|D_p^{2j}\phi_p(D_p)D_p^{m_2}s'\right\|_{L^2}\\ \leqslant& C_{l,m_1,m_2}p^{-l+2m_1}\left\|s'\right\|_{L^2}, \end{aligned} \tag{3.4.28}$$

where $[\cdot]$ denotes the integer part. Clearly,

$$\left\langle D_p^{m_2}\phi_p(D_p)Qs, s'\right\rangle = \left\langle s, Q^*\phi_p(D_p)D_p^{m_2}s'\right\rangle. \tag{3.4.29}$$

Combining (3.4.28) and (3.4.29) we get

$$\left\|D_p^{m_2}\phi_p(D_p)Qs\right\|_{L^2} \leqslant C_{l,m_1,m_2}p^{-l+2m_1}\left\|s\right\|_{L^2}. \tag{3.4.30}$$

If P, Q are differential operators of order m, m_1 with compact support in U_i, U_j, respectively. By (3.4.2) and (3.4.30),

$$\begin{aligned}\left\|P\phi_p(D_p)Qs\right\|_{L^2} \leqslant& C_m p^{2m+2}\sum_{j=0}^{[\frac{m}{2}]+1} p^{-2j}\left\|D_p^{2j}\phi_p(D_p)Qs\right\|_{L^2}\\ \leqslant& C_{l,m_1,m}p^{-l}\left\|s\right\|_{L^2}. \end{aligned} \tag{3.4.31}$$

From Sobolev inequalities and (3.4.31),

$$\begin{aligned}\left\|\phi_p(D_p)Qs\right\|_{C^m} &\leqslant C\left\|\phi_p(D_p)Qs\right\|_{m+n+1}\\ &\leqslant C\Big(\left\|P\phi_p(D_p)Qs\right\|_{L^2}+\left\|\phi_p(D_p)Qs\right\|_{L^2}\Big)\\ &\leqslant C_{l,m_1,m}p^{-l}\left\|s\right\|_{L^2}.\end{aligned} \tag{3.4.32}$$

On the other hand,

$$\phi_p(D_p)Qs(x)=\int_X \langle Q_y\big(\phi_p(D_p)(x,y)\big),s(y)\rangle dv_X(y)=\langle Q_\bullet\phi_p(D_p)(x,\cdot),s\rangle. \tag{3.4.33}$$

Then it is a consequence of (3.4.32) and (3.4.33) that for every $x\in X$,

$$\left\|Q_\bullet\phi_p(D_p)(x,\cdot)\right\|_{L^2}\leqslant C_{l,m}p^{-l}. \tag{3.4.34}$$

Since $\phi_p(D_p)(x,y)=\phi_p(D_p)(y,x)$, then the first inequality in (3.4.21) holds.

Note

$$F(D_p)(x,\cdot)=(\int_{-\infty}^{\infty}\theta(v)dv)^{-1}\int_{-\infty}^{\infty}\cos(vD_p)(x,\cdot)\theta(v)dv. \tag{3.4.35}$$

By the finite propagation speed of solutions of hyperbolic equations, $F(D_p)(x,y)$ only depends on the restriction of D_p to $B^X(x,\varepsilon)$, and equal zero if $d(x,y)\geqslant\varepsilon$. Thus we get the second inequality in (3.4.21). The proof of Proposition 3.16 is complete. □

3.4.2 Rescaling and Taylor expansion of the operator D_p^2

Fix $x_0\in X$. Set $B_a=B^{T_{x_0}X}(0,a)$ for $a>0$. We identify $B_{4\varepsilon}$ with $B^X(x_0,4\varepsilon)$ by the exponential map $Z\to\exp^X_{x_0}(Z)$ for $Z\in T_{x_0}X$. For $Z\in B_{4\varepsilon}$, we identify L_Z,E_Z and $\Lambda(T^{*(0,1)}_Z X)$ to L_{x_0},E_{x_0} and $\Lambda(T^{*(0,1)}_{x_0}X)$ by parallel transport with respect to the connection ∇^L,∇^E and $\nabla^{B,\Lambda^{0,\bullet}}$, respectively, along the curve $u\to uZ, u\in[0,1]$. Thus on $B_{4\varepsilon}$, $(L,h^L),(E,h^E)$, $(\Lambda(T^{*(0,1)}X),h^{\Lambda(T^{*(0,1)}X)})$ are identified to the trivial Hermitian bundles $(L_{x_0},h^{L_{x_0}})$, $(E_{x_0},\, h^{E_{x_0}})$, $(\Lambda(T^{*(0,1)}_{x_0}X),h^{\Lambda(T^{*(0,1)}_{x_0}X)})$. If Γ^L,Γ^E and $\Gamma^{B,\Lambda^{0,\bullet}}$ denote the corresponding connection form of ∇^L,∇^E and $\nabla^{B,\Lambda^{0,\bullet}}$ on $B_{4\varepsilon}$, respectively, then $\Gamma^L,\Gamma^E,\Gamma^{B,\Lambda^{0,\bullet}}$ are skew-adjoint with respect to $h^{L_{x_0}},h^{E_{x_0}},\ h^{\Lambda(T^{*(0,1)}_{x_0}X)}$. Moreover, (cf. [30, (1.2.30)])

$$\Gamma^\bullet_{x_0}=0,\quad\text{for }\Gamma^\bullet=\Gamma^L,\Gamma^E\text{ or }\Gamma^{B,\Lambda^{0,\bullet}}. \tag{3.4.36}$$

In view of (3.2.15) and (3.4.36) we get

$$\Gamma^{TX}_{x_0}+S^B_{x_0}=0. \tag{3.4.37}$$

Let $\{e_1,\dots,e_{2n}\}$ be an orthonormal basis of $T_{x_0}X$. The coordinates on $T_{x_0}X\simeq\mathbb{R}^{2n}$ is given by

$$(Z_1,\cdots,Z_{2n})\in\mathbb{R}^{2n}\to\sum_{j=1}^{2n}Z_je_j\in T_{x_0}X. \tag{3.4.38}$$

Let $\phi_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the map defined by $\phi_\varepsilon(Z) = \rho(\frac{|Z|}{\varepsilon})Z$ with

$$\rho(v) = 1, \text{ if } |v| \leqslant 2 \text{ and } \rho(v) = 0, \text{ if } |v| \geqslant 4. \tag{3.4.39}$$

Let $\mathcal{R}$ be the radial vector field defined by

$$\mathcal{R} = \sum_{j=1}^{2n} Z_j e_j = Z. \tag{3.4.40}$$

For the trivial vector bundle $E_0 := (E_{x_0}, h^{E_{x_0}})$, we define a Hermitian connection on $X_0 := T_{x_0}X$ by

$$\nabla^{E_0} = \nabla + \rho(\frac{|Z|}{\varepsilon})\Gamma^E. \tag{3.4.41}$$

For the trivial vector bundle $L_0 := (L_{x_0}, h^{L_{x_0}})$, we define a Hermitian connection on $X_0 := T_{x_0}X$ by

$$\nabla^{L_0} = \phi_\varepsilon^*(\nabla^L) + \frac{1}{2}\big(1 - \rho^2(\frac{|Z|}{\varepsilon})\big) R^L_{x_0}(\mathcal{R}, \cdot). \tag{3.4.42}$$

Then its curvature $R^{L_0} = (\nabla^{L_0})^2$ equals

$$\phi_\varepsilon^*(R^L) + \frac{1}{2}\Big[\phi_\varepsilon^*(\nabla^L), \big(1 - \rho^2(\frac{|Z|}{\varepsilon})\big) R^L_{x_0}(\mathcal{R}, \cdot)\Big]. \tag{3.4.43}$$

Clearly,

$$(\phi_\varepsilon)_* e_i = \rho(\frac{|Z|}{\varepsilon}) e_i + \rho'(\frac{|Z|}{\varepsilon}) \frac{Z_i}{\varepsilon|Z|}\mathcal{R}. \tag{3.4.44}$$

Therefore,

$$\begin{aligned}
&\big(\phi_\varepsilon^*(R^L)\big)_Z(e_i, e_j)\\
=&R^L_{\phi_\varepsilon(Z)}\big((\phi_\varepsilon)_* e_i, (\phi_\varepsilon)_* e_j\big)\\
=&\rho^2(\frac{|Z|}{\varepsilon}) R^L_{\phi_\varepsilon(Z)}\big(e_i, e_j\big) + \rho(\frac{|Z|}{\varepsilon})\rho'(\frac{|Z|}{\varepsilon})\Big(\sum_{k=1}^{2n} \frac{Z_k dZ_k}{\varepsilon|Z|} \wedge R^L_{\phi_\varepsilon(Z)}(\mathcal{R}, \cdot)\Big)(e_i, e_j).
\end{aligned}$$

That is

$$\phi_\varepsilon^*(R^L) = \rho^2(\frac{|Z|}{\varepsilon}) R^L_{\phi_\varepsilon(Z)} + (\rho\rho')(\frac{|Z|}{\varepsilon}) \sum_{k=1}^{2n} \frac{Z_k e^k}{\varepsilon|Z|} \wedge R^L_{\phi_\varepsilon(Z)}(\mathcal{R}, \cdot). \tag{3.4.45}$$

On the other hand,

$$\begin{aligned}
&\Big[\phi_\varepsilon^*(\nabla^L), \big(1 - \rho^2(\frac{|Z|}{\varepsilon})\big) R^L_{x_0}(\mathcal{R}, \cdot)\Big]\\
=&d\Big(1 - \rho^2(\frac{|Z|}{\varepsilon}) R^L_{x_0}(\mathcal{R}, \cdot)\Big)\\
=&-2(\rho\rho')(\frac{|Z|}{\varepsilon}) \sum_{k=1}^{2n} \frac{Z_k e^k}{\varepsilon|Z|} \wedge R^L_{x_0}(\mathcal{R}, \cdot) + 2\big(1 - \rho^2(\frac{|Z|}{\varepsilon})\big) R^L_{x_0}.
\end{aligned} \tag{3.4.46}$$

Substituting (3.4.45) and (3.4.46) into (3.4.43) we get

$$\begin{aligned} R^{L_0} =& \big(1-\rho^2(\frac{|Z|}{\varepsilon})\big)R^L_{x_0} + \rho^2(\frac{|Z|}{\varepsilon})R^L_{\phi_\varepsilon(Z)} \\ &- (\rho\rho')(\frac{|Z|}{\varepsilon})\sum_{k=1}^{2n}\frac{Z_k e^k}{\varepsilon|Z|}\wedge\Big(R^L_{x_0}(\mathcal{R},\cdot) - R^L_{\phi_\varepsilon(Z)}(\mathcal{R},\cdot)\Big). \end{aligned} \tag{3.4.47}$$

It is clear that $R^{L_0} = R^L_{x_0}$ for $|Z| \leqslant 2\varepsilon$. The third summand on the righthand side of (3.4.47) tends to zero as $|Z|$ tends to 0. Hence there exists $\eta > 0$ such that for any $0 < |Z| < \eta$ we have

$$\Big|(\rho\rho')(\frac{|Z|}{\varepsilon})\sum_{k=1}^{2n}\frac{Z_k e^k}{\varepsilon}\wedge\big(R^L_{x_0}(\frac{\mathcal{R}}{|Z|},\cdot) - R^L_{\phi_\varepsilon(Z)}(\frac{\mathcal{R}}{|Z|},\cdot)\big)\Big| \leqslant \frac{2}{5}\pi. \tag{3.4.48}$$

Then

$$R^{L_0}_Z(v_j,\overline{v}_j) \leqslant -\frac{8}{5}\pi, \text{ if } j\leqslant q \text{ and } R^L_Z(v_j,\overline{v}_j) \geqslant \frac{8}{5}\pi, \text{ if } j \geqslant q+1, \tag{3.4.49}$$

where $\big\{v_1,\ldots,v_n\big\}$ is a local orthonormal frame of $T^{(1,0)}X$ such that the relation (3.1.13) holds. In the sequel we fix $\varepsilon = \frac{1}{4}\eta$.

Let $g^{TX_0}(Z) = g^{TX}(\phi_\varepsilon(Z)), J_0(Z) = J(\phi_\varepsilon(Z))$ be the metric and the almost structure on X_0. Let $T^{*(0,1)}X_0$ be the anti-holomorphic cotangent bundle of (X_0, J_0). Then $T^{*(0,1)}_{Z,J_0}X_0$ is naturally identified with $T^{*(0,1)}_{\phi_\varepsilon(Z),J}X_0$. We identify $\Lambda(T^{*(0,1)}_Z X_0)$ with $\Lambda(T^{*(0,1)}_{x_0}X)$ by identifying first $\Lambda(T^{*(0,1)}_Z X_0)$ with $\Lambda(T^{*(0,1)}_{\phi_\varepsilon(Z),J}X_0)$ which in turn is identified with $\Lambda(T^{*(0,1)}_{x_0}X)$ by using the parallel transport with respect to the connection $\nabla^{B,\Lambda^{0,\bullet}}$ along the curve $t\to t\phi_\varepsilon(Z), t\in[0,1]$. We trivialized $\Lambda(T^{*(0,1)}X_0)$ in this way.

We trivialize the Hermitian line bundle $\det(T^{(1,0)}X_0)$ by identifying first $\det(T^{(1,0)}X_0)_Z$ to $\det(T^{(1,0)}X)_{\phi_\varepsilon(Z)}$, and then to $\det(T^{(1,0)}X)_{x_0}$ by using parallel transport along the curve $t\mapsto t\phi_\varepsilon(Z), t\in[0,1]$ with respect to the connection $\nabla^{\det(T^{(1,0)}X)}$. Let $\nabla^{\det_0}$ be the Hermitian connection on $\det(T^{(1,0)}X)_{x_0}$ defined by

$$\nabla^{\det_0} = d + \rho(\frac{|Z|}{\varepsilon})\Gamma^{\det(T^{(1,0)}X)}. \tag{3.4.50}$$

Let ∇^{TX_0} denote the Levi-Civita connection of (X_0, g^{TX_0}). Then g^{TX_0} and $\nabla^{\det_0}$ define a Clifford connection ∇^{Cl_0} on $\Lambda(T^{*(0,1)}X_0)$, i.e.,

$$\nabla^{Cl_0} = d + \frac{1}{4}\big\langle\Gamma^{TX_0}\tilde{e}_i,\tilde{e}_j\big\rangle c(\tilde{e}_i)c(\tilde{e}_j) + \rho(\frac{|Z|}{\varepsilon})\Gamma^{\det(T^{(1,0)}X)}, \tag{3.4.51}$$

where $\{\tilde{e}_1,\ldots,\tilde{e}_{2n}\}$ is a local orthonormal frame of (X_0, g^{TX_0}) and Γ^{TX_0} is the corresponding connection form of ∇^{TX_0} associated to the frame $\{\tilde{e}_1,\ldots,\tilde{e}_{2n}\}$.

Since g^{TX_0} is trivial outside $B_{4\varepsilon}$ and $\rho(\frac{|Z|}{\varepsilon}) = 0$ outside $B_{4\varepsilon}$, the connection form Γ^{Cl_0} of ∇^{Cl_0} associated to the above trivialization of $\Lambda(T^{*(0,1)}X_0)$ on $\mathbb{R}^{2n}$ satisfies

$$\Gamma^{Cl_0} = 0, \text{ for } |Z| \geqslant 4\varepsilon. \tag{3.4.52}$$

Denote by $D^c_{0,p}$ be the Spin^c Dirac operator on $\mathbb{R}^{2n}$ acting on $E_{0,p} = \Lambda(T^{*(0,1)}X_0) \otimes L_0^p \otimes E_0$, i.e.,

$$D^c_{0,p} = \sum_{j=1}^{2n} c(\tilde{e}_j)\nabla^{Cl_0\otimes L_0^p\otimes E_0}_{\tilde{e}_j}, \tag{3.4.53}$$

where $\nabla^{Cl_0\otimes L_0^p\otimes E_0}$ is the connection on $E_{0,p}$ induced by $\nabla^{Cl_0}, \nabla^{L_0}$ and ∇^{E_0}. Set

$$A_0 = -\frac{1}{4}\rho(\frac{|Z|}{\varepsilon})T_{as,Z}. \tag{3.4.54}$$

The modified Dirac operator on the almost complex manifold (X_0, J_0) is given by

$$D^{c,A_0}_p = D^c_{0,p} + {}^c(A_0), \tag{3.4.55}$$

which coincides with D_p on $B_{2\varepsilon}$. It is a consequence of (3.4.49) that the following spectral gap of the operator $(D^{c,A_0}_p)^2$ holds:

$$\operatorname{Spec}\big((D^{c,A_0}_p)^2\big) \subset \{0\} \cup [\frac{16\pi}{5}p - C, \infty). \tag{3.4.56}$$

Let $P_{0,p}$ be the orthogonal projection from $L^2(X_0, E_{0,p})$ onto $\operatorname{Ker}\big((D^{c,A_0}_p)^2\big)$ and let $P_{0,p}(x,y)$ be the smooth kernel of $P_{0,p}$ with respect to the volume form $dv_{X_0}(y)$. As a consequence of (3.4.56), the following Proposition is an analogue of Proposition 3.16 for the operator D^{c,A_0}_p.

Proposition 3.17. *For any $l, m \in \mathbb{N}$, there exist $C_{l,m} > 0$ such that for $x, y \in X$,*

$$\begin{aligned} &\big|F(D^{c,A_0}_p)(x,y) - P_{0,p}(x,y)\big|_{C^m} \leqslant C_{l,m}p^{-l}, \\ &\big|P_{0,p}(x,y)\big|_{C^m} \leqslant C_{l,m}p^{-l}, \text{ if } d(x,y) \geqslant \varepsilon. \end{aligned} \tag{3.4.57}$$

Note $F(D^{c,A_0}_p)(x,y) = F(D_p)(x,y)$ if $x, y \in B_{2\varepsilon}$. Combining Proposition 3.16 and Proposition 3.17, we get

Proposition 3.18. *For any $l, m \in \mathbb{N}$, there exist $C_{l,m} > 0$ such that for $x, y \in B_\varepsilon$,*

$$\big|P_{0,p}(x,y) - P_p(x,y)\big|_{C^m} \leqslant C_{l,m}p^{-l}. \tag{3.4.58}$$

Let s_L be a unit vector of L_{x_0}. Using s_L, we get an isometry $E^q_{p,x_0} \simeq \big(\Lambda^q(T^{*(0,1)}X) \otimes E\big)_{x_0} := \mathbb{E}_{x_0}$. As the operator $\big(D^{c,A_0}_p\big)^2$ takes values in $\operatorname{End}(E_{p,x_0}) = \operatorname{End}(\mathbb{E}_{x_0})$ under the natural identification $\operatorname{End}(L^p) \simeq \mathbb{C}$ (which does not depend on s_L), our formulas do not depend on the choice of s_L. Now under this identification, we will consider $\big(D^{c,A_0}_p\big)^2$ acting on $C^\infty(X_0, \mathbb{E}_{x_0})$.

Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $k(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = k(Z)dv_{TX}(Z) \tag{3.4.59}$$

with $k(0) = 1$. Let ∇^{A_0} be the connection induced by ∇^{Cl_0} and A_0 on $\Lambda(T^{*(0,1)}X_0)$ as before, i.e.,

$$\nabla^{A_0}_U = \nabla^{Cl_0}_U + {}^c(i_U A_0), \quad \text{for } U \in TX_0. \tag{3.4.60}$$

Let Γ^{A_0} be the corresponding connection form of ∇^{A_0}.

For $s \in C^\infty(\mathbb{R}^{2n}, \mathbb{E}_{x_0}), Z \in \mathbb{R}^{2n}$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned} \big(\delta_t s\big)(Z) = s(\frac{Z}{t}), \quad \nabla_{t,\bullet} = \delta_t^{-1} t k^{\frac{1}{2}} \nabla^{A_0} k^{-\frac{1}{2}} \delta_t, \\ \mathcal{L}_2^t = \delta_t^{-1} t^2 k^{\frac{1}{2}} \big(D_p^{c,A_0}\big)^2 k^{-\frac{1}{2}} \delta_t. \end{aligned} \tag{3.4.61}$$

We restate the following expansion of the operator $\mathcal{L}_2^t$, see [30, Theorem 4.1.7].

Theorem 3.19. *There exist polynomials $A_{ij,r}$ (resp. $B_{i,r}, C_r$) ($r \in \mathbb{N}, i, j \in \{1, \cdots, 2n\}$) in Z with the following properties:*
(1) their coefficients are polynomials in R^{TX} (resp. $R^{TX}, R^{B,\Lambda^{0,\bullet}}, R^E, R^{\det(T^{(1,0)}X)}, d\Theta, R^L$) and their derivatives at x_0 up to order $r-2$ (resp. $r-2, r-2, r-2, r-2, r-1, r$),
(2) $A_{ij,r}$ is a homogenous polynomial in Z of degree r, the degree in Z of $B_{i,r}$ is $\leqslant r+1$ (resp. of C_r is $\leqslant r+2$), and has the same parity as $r-1$ (resp. r),
(3) if we denote

$$\mathcal{O}_r = A_{ij,r} \nabla_{e_i} \nabla_{e_j} + B_{i,r} \nabla_{e_i} + C_r, \tag{3.4.62}$$

then

$$\mathcal{L}_2^t = \mathcal{L}_2^0 + \sum_{r=1}^m t^r \mathcal{O}_r + O(t^{m+1}), \tag{3.4.63}$$

where

$$\mathcal{L}_2^0 = -(\nabla_{0,e_i})^2 - 2n\pi - 2w_{d,x_0}, \ \nabla_{0,\bullet} = \nabla_\bullet + \frac{1}{2} R^L_{x_0}(\mathcal{R}, \cdot). \tag{3.4.64}$$

Moreover, there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}, t \leqslant 1$, the derivatives of order $\leqslant k$ of the coefficients of the operator $O(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$.

Proof. We give the proof for the sake of completeness. We will add a subscript 0 to indicate the corresponding object on X_0, e.g., $R^{E_0}, R^{\det_0}$ are the curvatures of $\nabla^{E_0}, \nabla^{\det_0}$. As in (3.2.13), let Φ_{E_0} be the smooth self-adjoint section of $\mathrm{End}(\mathbb{E}_{x_0})$ on X_0,

$$\Phi_{E_0} = \frac{r^{X_0}}{4} + {}^c\big(R^{E_0} + \frac{1}{2} R^{\det_0}\big) + {}^c(dA_0) - 2|A_0|^2. \tag{3.4.65}$$

Then Lichnerowicz formula (3.2.12) entails

$$\big(D_p^{c,A_0}\big)^2 = \Delta^{A_0} + p\, {}^c(R^{L_0}) + \Phi_{E_0}. \tag{3.4.66}$$

Set

$$g_{ij}(Z) = g^{TX_0}(e_i, e_j)(Z), \quad \nabla^{TX_0}_{e_i} e_j = \Gamma^k_{ij} e_l. \tag{3.4.67}$$

Let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. By (3.4.61),

$$\begin{aligned}\nabla_{t,e_i} =& \nabla_{e_i} + \Big(\frac{1}{t}\Gamma^{L_0}_{tZ}(e_i) + t\Gamma^{E_0}_{tZ}(e_i) + \Gamma^{A_0}_{tZ}(e_i)\Big) - \frac{t}{2}\big(k^{-1}\nabla_{e_i}k\big)(tZ),\\ \mathcal{L}^t_2 =& - g^{ij}(tZ)\Big(\nabla_{t,e_i}\nabla_{t,e_j} - \Gamma^k_{ij}\nabla_{t,e_k}\Big) + \ {}^c(R^{L_0}_{tZ}) + t^2\Phi_{E_0,tZ}.\end{aligned} \tag{3.4.68}$$

It is a consequence of [30, (1.2.27) and (1.2.29)] that

$$g_{ij}(tZ) = \delta_{ij} + \sum_{r=2}^{m} t^r D_{ij,r}(Z) + O(t^{m+1}), \tag{3.4.69}$$

where $D_{ij,r}$ is polynomial in R^{TX} and its derivatives at x_0 up to order $r-2$ and $D_{ij,r}$ is homogenous polynomial in Z of degree r, e.g.,

$$D_{ij,2} = \frac{1}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\Big\rangle, \ D_{ij,3} = -\frac{1}{6}\Big\langle \big(\nabla^X_{\mathcal{R}} R^{TX}\big)_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\Big\rangle. \tag{3.4.70}$$

Then

$$k(tZ) = \det{}^{\frac{1}{2}}(g_{ij}(tZ)) = 1 + \frac{1}{2}\sum_{r=2}^{m} t^r D_{ii,r}(Z) + O(t^{m+1}), \tag{3.4.71}$$

and

$$\begin{aligned}t(k^{-1}\nabla_{e_i}k)(tZ) =& k^{-1}\nabla_{e_i}\big(k(tZ)\big)\\ =& \frac{1}{2}\sum_{r=2}^{m} t^r\nabla_{e_i}D_{jj,r}(Z) - \frac{1}{4}\sum_{r,l=2}^{m} t^{r+l}D_{jj,r}\big(\nabla_{e_i}D_{kk,l}\big) + O(t^{m+1}).\end{aligned} \tag{3.4.72}$$

By [30, (1.2.30)],

$$\begin{aligned}\nabla_{t,e_i} =& \nabla_{e_i} + \sum_{r=1}^{m+1}\frac{t^{r-1}}{r+1}\sum_{|\alpha|=r-1}\big(\partial^\alpha R^L\big)_{x_0}(\mathcal{R}, e_i)\frac{Z^\alpha}{\alpha!}\\ &+ \sum_{r=1}^{m-1}\frac{t^{r+1}}{r+1}\sum_{|\alpha|=r-1}\big(\partial^\alpha R^E + \partial^\alpha R^{B,\Lambda^{0,\bullet}}\big)_{x_0}(\mathcal{R}, e_i)\frac{Z^\alpha}{\alpha!}\\ &+ \frac{1}{2}\sum_{r=2}^{m} t^r\nabla_{e_i}D_{jj,r}(Z) - \frac{1}{4}\sum_{r,l=2}^{m} t^{r+l}D_{jj,r}\big(\nabla_{e_i}D_{kk,l}\big) + O(t^{m+1})\\ =& \nabla_{0,e_i} + \sum_{r=1}^{m} t^m\tilde{D}_{i,r}(Z) + O(t^{m+1}),\end{aligned} \tag{3.4.73}$$

where $\tilde{D}_{i,r}(Z)$ satisfies
(i) polynomial in $R^{TX}, R^{B,\Lambda^{0,\bullet}}, R^E, R^{\det(T^{(1,0)}X)}, d\Theta, R^L$ and their derivatives at x_0 up to order $r-2, r-2, r-2, r-2, r-1$ and r, respectively;
(ii) polynomial in Z of degree $\leqslant r+1$ and has the same parity as $r-1$.

It is known (cf. [30, (4.1.102)]) that

$$\Gamma_{ij}^k(tZ) = \frac{1}{2} g^{kl}(tZ)\Big(\nabla_{e_j} g_{il} + \nabla_{e_i} g_{jl} - \nabla_{e_l} g_{ij}\Big)(tZ). \tag{3.4.74}$$

Then $\Gamma_{ij}^k(tZ)$ has similar expression as (3.4.72).

Clearly,

$$^c(R_{tZ}^{L_0}) = \frac{1}{2} R_{tZ}^{L_0}(\tilde{e}_i, \tilde{e}_j) c(\tilde{e}_i) c(\tilde{e}_j) = {}^c(R_{x_0}^L) + \sum_{r=1}^m t^r D_r(Z) + O(t^{m+1}), \tag{3.4.75}$$

where $D_r(Z)$ is polynomial in R^L and its derivatives at x_0 up to order r and $D_r(Z)$ is homogenous polynomial in Z of degree r, e.g.,

$$\begin{aligned} D_1 =& \big(\nabla_{\mathcal{R}}^B R^L\big)_{x_0}(e_i, e_j) c(e_i) c(e_j), \\ D_2 =& \frac{1}{2}\big(\nabla^B \nabla^B R^L\big)_{(\mathcal{R},\mathcal{R}),x_0}(e_i, e_j) c(e_i) c(e_j). \end{aligned} \tag{3.4.76}$$

Substituting (3.4.73), (3.4.74) and (3.4.75) into (3.4.68) we get

$$\mathcal{L}_2^t = \mathcal{L}_2^0 + \sum_{r=1}^m t^r \mathcal{O}_r + O(t^{m+1}), \tag{3.4.77}$$

where

$$\mathcal{O}_r = -D_{ij,r} \nabla_{e_i} \nabla_{e_j} + B_{i,r} \nabla_{e_i} + C_r \tag{3.4.78}$$

with $B_{i,r}$ and C_r satisfying the conditions (1) and (2) of Theorem 3.19. The proof of Theorem 3.19 is complete. □

3.4.3 Sobolev estimate on the resolvent $(\lambda - \mathcal{L}_2^t)^{-1}$

Let $h_0^{\Lambda^{0,\bullet}}$ be the Hermitian metric on $\Lambda(T^{*(0,1)}X_0)$ induced by g^{TX_0} and J_0. By the trivialization of $\Lambda(T^{*(0,1)}X_0)$, $(\Lambda(T^{*(0,1)}X_0), h_0^{\Lambda^{0,\bullet}})$ is identified to the trivial Hermitian vector bundle $(\Lambda(T_{x_0}^{*(0,1)}X), h^{\Lambda_{x_0}^{0,\bullet}})$.

Let $h^{\mathbb{E}_{x_0}}$ be the metric on $\mathbb{E}_{x_0}$ induced by $h^{\Lambda_{x_0}^{0,\bullet}}$ and $h^{E_{x_0}}$. We denote by $\langle\cdot,\cdot\rangle_{0,L^2}$ and $\|\cdot\|_{0,L^2}$ the scalar product and the L^2-norm on $C^\infty(X_0, \Lambda(T^{*(0,1)}X_0)\otimes E_0)$ induced by $h_0^{\Lambda^{0,\bullet}}, h^{E_0}$. Then $\langle\cdot,\cdot\rangle_{0,L^2}$ is the same as the scalar product on $C^\infty(X_0, \mathbb{E}_{x_0})$ induced by $h^{\mathbb{E}_{x_0}}, dv_{X_0}$ under our trivialization.

For $s \in C^\infty(T_{x_0}X, \mathbb{E}_{x_0})$, set

$$\begin{aligned}\|s\|_{t,0}^2 :=& \|s\|_0^2 = \int_{\mathbb{R}^{2n}} |s(Z)|^2 dv_{TX}(Z),\\ \|s\|_{t,m}^2 =& \sum_{l=0}^{m} \sum_{i_1,\cdots,i_l=1}^{2n} \|\nabla_{t,e_{i_1}} \cdots \nabla_{t,e_{i_l}} s\|_0^2.\end{aligned} \tag{3.4.79}$$

We denote by $\langle \cdot,\cdot \rangle_{t,0}$ the inner product on $C^\infty(X_0, \mathbb{E}_{x_0})$ corresponding to $\|\cdot\|_{t,0}^2$. Let $\mathbf{H}_t^m$ be the Sobolev space of order m with norm $\|\cdot\|_{t,m}$. Let $\mathbf{H}_t^{-1}$ be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on $\mathbf{H}_t^{-1}$ defined by

$$\|s\|_{t,-1} = \sup_{0\neq s_1 \in \mathbf{H}_t^1} \frac{|\langle s, s_1\rangle|}{\|s_1\|_{t,1}}. \tag{3.4.80}$$

If $A \in \mathcal{L}(\mathbf{H}_t^m, \mathbf{H}_t^{m'})$, we denote by $\|A\|_t^{m,m'}$ the norm of A with respect to the norm $\|\cdot\|_{t,m}$ and $\|\cdot\|_{t,m'}$.

Remark 3.20. Since D_p^{c,A_0} is symmetric with respect to $\|\cdot\|_{0,L^2}$, by (3.4.61), $\mathcal{L}_2^t$ is a formal adjoint elliptic operator with respect to $\|\cdot\|_0$, i.e., for $s_1, s_2 \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,

$$\begin{aligned}\langle \mathcal{L}_2^t s_1, s_2\rangle_{t,0} =& \int_{\mathbb{R}^{2n}} \Big\langle \Big(\delta_t^{-1} t^2 k^{\frac12} \big(D_p^{c,A_0}\big)^2 k^{-\frac12} \delta_t s_1\Big)(Z), s_2(Z)\Big\rangle dv_{TX}(Z)\\ =& \int_{\mathbb{R}^{2n}} \Big\langle \Big(\big(D_p^{c,A_0}\big)^2 k^{-\frac12}\delta_t s_1\Big)(tZ), k^{-\frac12}(tZ) s_2(Z)\Big\rangle k(tZ) dv_{TX}(Z)\\ =& \langle s_1, \mathcal{L}_2^t s_2\rangle_{t,0}.\end{aligned} \tag{3.4.81}$$

Theorem 3.21. *There exist constants $C_1, C_2, C_3 > 0$ such that for $t \in (0,1]$ and any $s_1, s_2 \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,*

$$\begin{aligned}\langle \mathcal{L}_2^t s_1, s_1\rangle_{t,0} \geqslant& C_1\|s_1\|_{t,1}^2 - C_2\|s_1\|_{t,0}^2;\\ \Big|\langle \mathcal{L}_2^t s_1, s_2\rangle_{t,0}\Big| \leqslant& C_3\|s_1\|_{t,1}\|s_2\|_{t,1}.\end{aligned} \tag{3.4.82}$$

Proof. Clearly,

$$\begin{aligned}&\Big\langle \delta_t^{-1} t^2 k^{\frac12} \Delta^{A_0} k^{-\frac12}\delta_t s_1, s_1\Big\rangle_{t,0} \qquad (3.4.83)\\ =& t^2 \int_{\mathbb{R}^{2n}} \Big\langle \Big(\delta_t^{-1} t^2 k^{\frac12}\Delta^{A_0} k^{-\frac12}\delta_t s_1\Big)(Z), s_1(Z)\Big\rangle dv_{TX}(Z)\\ =& t^{2-2n}\int_{\mathbb{R}^{2n}} \Big\langle \big(\Delta^{A_0} k^{-\frac12}\delta_t s_1\big)(Z), \big(k^{-\frac12}\delta_t s_1\big)(Z)\Big\rangle dv_{X_0}(Z)\\ =& t^{2-2n}\int_{\mathbb{R}^{2n}} \Big|\big(k^{\frac12}\nabla^{A_0} k^{-\frac12}\delta_t s_1\big)(Z)\Big|^2 dv_{X_0}(Z)\\ =& \int_{\mathbb{R}^{2n}} \Big|\big(\delta_t^{-1} t k^{\frac12}\nabla^{A_0} k^{-\frac12}\delta_t s_1\big)(Z)\Big|^2 dv_{X_0}(Z)\\ =& \|\nabla_t s_1\|_{L^2}^2.\end{aligned}$$

In view of (3.4.61), (3.4.66) and (3.4.83) we get

$$\big\langle \mathcal{L}_2^t s_1, s_1\big\rangle_{t,0} = \big\|\nabla_t s_1\big\|_{L^2}^2 + \Big\langle \big({}^c(R_{tZ}^{L_0}) + t^2\Phi_{E_0,tZ}\big)s_1, s_1\Big\rangle_{t,0}, \tag{3.4.84}$$

which implies the first inequality of (3.4.82). Similarly we have

$$\big\langle \mathcal{L}_2^t s_1, s_2\big\rangle_{t,0} = \big\langle \nabla_{t,\cdot} s_1, \nabla_{t,\cdot} s_2\big\rangle_{t,0} + \Big\langle \big({}^c(R_{tZ}^{L_0}) + t^2\Phi_{E_0,tZ}\big)s_1, s_2\Big\rangle_{t,0}. \tag{3.4.85}$$

Then the second inequality of (3.4.82) follows from (3.4.85). The proof of Theorem 3.21 is complete. □

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\frac{\pi}{2}$, and let γ be the oriented path in $\mathbb{C}$ consisting of the ray from $(+\infty, i)$ to (π, i), the interval from (π, i) to $(\pi, -i)$ and the ray from $(\pi, -i)$ to $(+\infty, -i)$.

It is an consequence of (3.4.61) that

$$\operatorname{Spec}(\mathcal{L}_2^t) = t^2 \operatorname{Spec}\big(D_p^{c,A_0}\big)^2. \tag{3.4.86}$$

In view of (3.4.56), there exist $t_0 \in (0,1]$ such that for $t \in (0, t_0]$,

$$\operatorname{Spec}(\mathcal{L}_2^t) \subset \{0\} \cup [2\pi, \infty). \tag{3.4.87}$$

Thus, $(\lambda - \mathcal{L}_2^t)^{-1}$ exists for $\lambda \in \delta \cup \gamma$.

Theorem 3.22. *There exists $C > 0$ such that for $t \in (0, t_0], \lambda \in \delta \cup \gamma$,*

$$\begin{aligned} &\big\|(\lambda - \mathcal{L}_2^t)^{-1}\big\|_t^{0,0} \leqslant C; \\ &\big\|(\lambda - \mathcal{L}_2^t)^{-1}\big\|_t^{-1,1} \leqslant C(1 + |\lambda|^2). \end{aligned} \tag{3.4.88}$$

Proof. Set

$$\operatorname{Dom}(\mathcal{L}_2^t) = \big\{ s \in \mathbf{H}_t^0, \quad \mathcal{L}_2^t s \in \mathbf{H}_t^0 \big\}, \tag{3.4.89}$$

where $\mathcal{L}_2^t s$ is calculated in the sense of distribution.

By (3.4.87), for every $s \in C_0^\infty(\mathbb{R}^{2n}, \mathbb{E}_{x_0}), \lambda \in \delta \cup \gamma$,

$$\big\|(\lambda - \mathcal{L}_2^t)s\big\|_{t,0}^2 \geqslant \big\|s\big\|_{t,0}^2. \tag{3.4.90}$$

Now we claim that

$$\lambda - \mathcal{L}_2^t : \ \operatorname{Dom}(\mathcal{L}_2^t) \to L^2(X_0, \mathbb{E}_{x_0}) \text{ is bijective for } \lambda \in \delta \cup \gamma. \tag{3.4.91}$$

First it follows from (3.4.90) that $\lambda - \mathcal{L}_2^t$ is injective. Given $s_j \in \operatorname{Dom}(\mathcal{L}_2^t)$, $(\lambda - \mathcal{L}_2^t)s_j \to v \in L^2(X_0, \mathbb{E}_{x_0})$, then

$$\big\|(\lambda - \mathcal{L}_2^t)(s_j - s_k)\big\|_{t,0}^2 \geqslant \big\|s_j - s_k\big\|_{t,0}^2. \tag{3.4.92}$$

Therefore there exists $s \in L^2(X_0, \mathbb{E}_{x_0})$ such that

$$s_j \to s \text{ in } L^2(X_0, \mathbb{E}_{x_0}). \tag{3.4.93}$$

For any $\phi \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,

$$\begin{aligned}\big\langle(\lambda - \mathcal{L}_2^t)s, \phi\big\rangle =& \big\langle s, (\lambda - \mathcal{L}_2^t)\phi\big\rangle \\ =& \lim_{j\to\infty} \big\langle s_j, (\lambda - \mathcal{L}_2^t)\phi\big\rangle \\ =& \lim_{j\to\infty} \big\langle(\lambda - \mathcal{L}_2^t)s_j, \phi\big\rangle = \big\langle v, \phi\big\rangle.\end{aligned} \tag{3.4.94}$$

That is

$$(\lambda - \mathcal{L}_2^t)s = v \quad \text{in } L^2(X_0, \mathbb{E}_{x_0}). \tag{3.4.95}$$

In particular, $s \in \mathrm{Dom}(\mathcal{L}_2^t)$ and $v \in \mathrm{Im}(\lambda - \mathcal{L}_2^t)$. Therefore, the image $\mathrm{Im}(\lambda - \mathcal{L}_2^t)$ is closed in $L^2(X_0, \mathbb{E}_{x_0})$. To complete the proof of (3.4.91), it suffice to prove

$$\mathrm{Im}(\lambda - \mathcal{L}_2^t) = L^2(X_0, \mathbb{E}_{x_0}). \tag{3.4.96}$$

If not, there exists a nonzero element $s \in L^2(X_0, \mathbb{E}_{x_0})$ such that s is orthogonal to $\mathrm{Im}(\lambda - \mathcal{L}_2^t)$. In particular, for any $s' \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,

$$0 = \big\langle s, (\lambda - \mathcal{L}_2^t)s'\big\rangle = \big\langle(\lambda - \mathcal{L}_2^t)s, s'\big\rangle. \tag{3.4.97}$$

Then we get

$$(\lambda - \mathcal{L}_2^t)s = 0. \tag{3.4.98}$$

Hence $s = 0$. This is a contradiction to the choice of s. Thus the claim (3.4.91) holds.

From (3.4.90) we get

$$\big\|(\lambda - \mathcal{L}_2^t)^{-1}\big\|_t^{0,0} \leqslant 1. \tag{3.4.99}$$

By (3.4.82) and (3.4.87), for $\lambda_0 \in \mathbb{R}, \lambda_0 \leqslant -\frac{\pi}{2}$, $(\lambda_0 - \mathcal{L}_2^t)^{-1}$ exists and

$$\big\|(\lambda_0 - \mathcal{L}_2^t)^{-1}\big\|_t^{-1,1} \leqslant \frac{1}{C_1}. \tag{3.4.100}$$

Clearly,

$$(\lambda - \mathcal{L}_2^t)^{-1} = (\lambda_0 - \mathcal{L}_2^t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathcal{L}_2^t)^{-1}(\lambda_0 - \mathcal{L}_2^t)^{-1}. \tag{3.4.101}$$

Combining (3.4.99), (3.4.100) and (3.4.101) we obtain

$$\begin{aligned}\big\|(\lambda - \mathcal{L}_2^t)^{-1}\big\|_t^{-1,0} \leqslant& \big\|(\lambda_0 - \mathcal{L}_2^t)^{-1}\big\|_t^{-1,0} + |\lambda - \lambda_0| \cdot \big\|(\lambda - \mathcal{L}_2^t)^{-1}\big\|_t^{0,0} \cdot \big\|(\lambda_0 - \mathcal{L}_2^t)^{-1}\big\|_t^{-1,0} \\ \leqslant& \frac{1}{C_1}\big(1 + |\lambda - \lambda_0|\big).\end{aligned} \tag{3.4.102}$$

Similarly,

$$(\lambda-\mathcal{L}_2^t)^{-1}=(\lambda_0-\mathcal{L}_2^t)^{-1}-(\lambda-\lambda_0)(\lambda_0-\mathcal{L}_2^t)^{-1}(\lambda-\mathcal{L}_2^t)^{-1}. \tag{3.4.103}$$

Now (3.4.99), (3.4.100), (3.4.102) and (3.4.103) entail

$$\begin{aligned}\left\|(\lambda-\mathcal{L}_2^t)^{-1}\right\|_t^{-1,1}\leqslant&\left\|(\lambda_0-\mathcal{L}_2^t)^{-1}\right\|_t^{-1,1}+|\lambda-\lambda_0|\cdot\left\|(\lambda_0-\mathcal{L}_2^t)^{-1}\right\|_t^{0,1}\cdot\left\|(\lambda-\mathcal{L}_2^t)^{-1}\right\|_t^{-1,0}\\ \leqslant&\frac{1}{C_1}+\frac{|\lambda-\lambda_0|}{C_1}(1+|\lambda-\lambda_0|).\end{aligned} \tag{3.4.104}$$

Now the second inequality of (3.4.88) follows immediately from (3.4.104). The proof of Theorem 3.22 is complete. □

Proposition 3.23. *Take $m\in\mathbb{N}$, there exists $C_m>0$ such that for $t\in(0,1]$, $Q_1,\cdots,Q_m\in\left\{\nabla_{t,e_i},Z_i\right\}_{i=1}^{2n}$ and $s_1,s_2\in C_0^\infty(\mathbb{R}^{2n},\mathbb{E}_{x_0})$,*

$$\left|\left\langle\left[Q_1,[Q_2,\cdots,[Q_m,\mathcal{L}_2^t]\cdots]\right]s_1,s_2\right\rangle_{t,0}\right|\leqslant C_m\left\|s_1\right\|_{t,1}\left\|s_2\right\|_{t,1}. \tag{3.4.105}$$

Proof. For every $s\in C^\infty(\mathbb{R}^{2n},\mathbb{E}_{x_0})$,

$$\begin{aligned}\left[\nabla_{t,e_i},Z_j\right]s=&\Big(\delta_t^{-1}tk^{\frac12}\nabla_{e_i}^{A_0}k^{-\frac12}\delta_tZ_js\Big)(Z)-Z_j\Big(\delta_t^{-1}tk^{\frac12}\nabla_{e_i}^{A_0}k^{-\frac12}\delta_ts\Big)(Z)\\ =&tk^{\frac12}(tZ)\Big(\nabla_{e_i}^{A_0}k^{-\frac12}\delta_t\big(Z_js\big)\Big)(tZ)-Z_jk^{\frac12}\nabla_{e_i}^{A_0}\big(k^{-\frac12}(tZ)s(Z)\big)\\ =&tk^{\frac12}(tZ)\Big(\nabla_{e_i}^{A_0}\frac{Z_j}{t}k^{-\frac12}\delta_ts\Big)(tZ)-Z_jk^{\frac12}\nabla_{e_i}^{A_0}\big(k^{-\frac12}(tZ)s(Z)\big)\\ =&tk^{\frac12}(tZ)\delta_{ij}\cdot\frac1tk^{-\frac12}(tZ)s(Z)\\ =&\delta_{ij}s(Z).\end{aligned} \tag{3.4.106}$$

That is

$$\left[\nabla_{t,e_i},Z_j\right]=\delta_{ij}. \tag{3.4.107}$$

Combining (3.4.68) and (3.4.107), we get

$$\left[\mathcal{L}_2^t,Z_j\right]=-2g^{ij}(tZ)\nabla_{t,e_i}+tg^{ij}(tZ)\Gamma_{ik}^j(tZ). \tag{3.4.108}$$

It follows from (3.4.108) that $\left[Z_j,\mathcal{L}_2^t\right]$ satisfies (3.4.105).

By (3.4.68) we get

$$\left[\nabla_{t,e_i},\nabla_{t,e_j}\right]=\left(R_{tZ}^{L_0}+t^2R_{tZ}^{A_0}+t^2R_{tZ}^{E_0}\right)(e_i,e_j). \tag{3.4.109}$$

Combining (3.4.68) and (3.4.109) we find that $\left[\nabla_{t,e_k},\mathcal{L}_2^t\right]$ has the same structure as $\mathcal{L}_2^t$ for $t\in(0,1]$, i.e., $\left[\nabla_{t,e_k},\mathcal{L}_2^t\right]$ is of the type

$$\sum_{ij}a_{ij}(t,tZ)\nabla_{t,e_i}\nabla_{t,e_j}+\sum_i d_i(t,tZ)\nabla_{t,e_i}+c(t,tZ), \tag{3.4.110}$$

where $a_{ij}(t,tZ), d_i(t,tZ), c(t,tZ)$ and their derivatives on Z are uniformly bounded for $Z \in \mathbb{R}^{2n}, t \in [0,1]$ and they are polynomials in t.

For $s_1, s_2 \in C_0^\infty(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, by integration by parts,

$$\begin{aligned}\Big\langle \nabla_{t,e_i} s_1, s_2 \Big\rangle_{t,0} =& \int_{\mathbb{R}^{2n}} \Big\langle \big(\delta_t^{-1} t k^{\frac{1}{2}} \nabla_{e_i}^{A_0} k^{-\frac{1}{2}} \delta_t s_1\big)(Z), s_2(Z) \Big\rangle dv_{TX}(Z) \\ =& t^{1-2n} \int_{\mathbb{R}^{2n}} \Big\langle \big(\nabla_{e_i}^{A_0} k^{-\frac{1}{2}} \delta_t s_1\big)(Z), \big(k^{-\frac{1}{2}} \delta_t s_2\big)(Z) \Big\rangle dv_{X_0}(Z) \\ =& t^{1-2n} \int_{\mathbb{R}^{2n}} e_i \big\langle k^{-\frac{1}{2}} \delta_t s_1, k^{-\frac{1}{2}} \delta_t s_2 \big\rangle dv_{X_0}(Z) - \Big\langle s_1, \nabla_{t,e_i} s_2 \Big\rangle \\ =& \int_{\mathbb{R}^{2n}} e_i \big\langle s_1, s_2 \big\rangle dv_{TX}(Z) - \Big\langle s_1, \nabla_{t,e_i} s_2 \Big\rangle \\ & - \int_{\mathbb{R}^{2n}} t\big(k^{-1} \nabla_{e_i} k\big)(tZ) \big\langle s_1, s_2 \big\rangle dv_{TX}(Z) \\ =& - \Big\langle s_1, \big(\nabla_{t,e_i} + t(k^{-1} \nabla_{e_i} k)(tZ)\big) s_2 \Big\rangle_{t,0}. \end{aligned} \tag{3.4.111}$$

Denote by $(\nabla_{t,e_i})^*$ the formal adjoint of ∇_{t,e_i} with respect to the product $\big\langle \cdot, \cdot \big\rangle_{t,0}$. Then (3.4.111) implies

$$(\nabla_{t,e_i})^* = -\nabla_{t,e_i} - t(k^{-1} \nabla_{e_i} k)(tZ), \tag{3.4.112}$$

and the last term of (3.4.112) and its derivatives in Z are uniformly bounded in $Z \in \mathbb{R}^{2n}, t \in [0,1]$.

By (3.4.110) and (3.4.112), (3.4.105) is verified for $m = 1$. By iteration, the expression $\big[Q_1, [Q_2, \cdots, [Q_m, \mathcal{L}_2^t] \cdots]\big]$ has the same structure (3.4.110) as $\mathcal{L}_2^t$. By (3.4.112), we get Proposition 3.23. □

Theorem 3.24. *For any $t \in (0, t_0], \lambda \in \delta \cup \gamma, m \in \mathbb{N}$, the resolvent $(\lambda - \mathcal{L}_2^t)^{-1}$ maps $\mathbf{H}_t^m$ into $\mathbf{H}_t^{m+1}$. Moreover, for any $\alpha \in \mathbb{N}^{2n}$, there exist $N \in \mathbb{N}, C_{\alpha,m} > 0$ such that for $t \in (0, t_0], \lambda \in \delta \cup \gamma, s \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,*

$$\Big\| Z^\alpha (\lambda - \mathcal{L}_2^t)^{-1} s \Big\|_{t,m+1} \leqslant C_{\alpha,m} (1 + |\lambda|^2)^N \sum_{\beta \leqslant \alpha} \big\| Z^\beta s \big\|_{t,m}. \tag{3.4.113}$$

Proof. For $Q_1, \cdots, Q_m \in \big\{\nabla_{t,e_i}\big\}_{i=1}^{2n}, Q_{m+1}, \cdots, Q_{m+|\alpha|} \in \big\{Z_i\big\}_{i=1}^{2n}$, we can express the expression $Q_1 \cdots Q_{m+|\alpha|} (\lambda - \mathcal{L}_2^t)^{-1}$ as a linear combination of operators of the type

$$\big[Q_{i_1}, [Q_{i_2}, \cdots, [Q_{i_{m'}}, (\lambda - \mathcal{L}_2^t)^{-1}] \cdots]\big] Q_{i_{m'+1}} \cdots Q_{i_{m+|\alpha|}}, \tag{3.4.114}$$

where $\big\{i_1, \cdots, i_{m+|\alpha|}\big\}$ is a perturbation of $\big\{1, \cdots, m + |\alpha|\big\}$.

Let $\mathcal{R}_t$ be the family of operators

$$\mathcal{R}_t = \Big\{ \big[Q_{j_1}, [Q_{j_2}, \cdots, [Q_{j_l}, \mathcal{L}_2^t] \cdots]\big] \Big\}. \tag{3.4.115}$$

If $\big[Q, \mathcal{L}_2^t\big] = S \in \mathcal{R}_t$, then

$$\big[Q, (\lambda - \mathcal{L}_2^t)^{-1}\big] = (\lambda - \mathcal{L}_2^t)^{-1} S (\lambda - \mathcal{L}_2^t)^{-1}. \tag{3.4.116}$$

By (3.4.116), any commutator $\big[Q_{i_1}, [Q_{i_2}, \cdots, [Q_{i_{m'}}, (\lambda - \mathcal{L}_2^t)^{-1}] \cdots]\big]$ is a linear combination of operators of the form

$$(\lambda - \mathcal{L}_2^t)^{-1} R_1 (\lambda - \mathcal{L}_2^t)^{-1} R_2 \cdots R_{m'} (\lambda - \mathcal{L}_2^t)^{-1} \tag{3.4.117}$$

with $R_1, \cdots, R_{m'} \in \mathcal{R}_t$.

By Proposition 3.23, for $R \in \mathcal{R}_t$,

$$\Big|\big\langle R s_1, s_2\big\rangle_{t,0}\Big| \leqslant C \big\|s_1\big\|_{t,1} \big\|s_2\big\|_{t,1}. \tag{3.4.118}$$

If $s_2 \neq 0$, then

$$\frac{\Big|\big\langle R s_1, s_2\big\rangle_{t,0}\Big|}{\big\|s_2\big\|_{t,1}} \leqslant C \big\|s_1\big\|_{t,1}. \tag{3.4.119}$$

It follows from (3.4.119) that

$$\big\|R\big\|_t^{1,-1} = \sum_{\|s\|_{t,1}=1} \big\|Rs\big\|_{t,-1} \leqslant C. \tag{3.4.120}$$

By Theorem 3.22, there exist $C > 0$ and $N \in \mathbb{N}$ such that the norm of $\|\cdot\|_t^{0,1}$ of the operators (3.4.117) is dominated by $C(1 + |\lambda|^2)^N$. □

Remark 3.25. Take $|\alpha| = 1, m = 0$ in (3.4.113) for example. Clearly,

$$\begin{aligned} \nabla_{t,e} Z_i (\lambda - \mathcal{L}_2^t)^{-1} =& (\lambda - \mathcal{L}_2^t)^{-1} \nabla_{t,e} Z_i + \big[\nabla_{t,e}, (\lambda - \mathcal{L}_2^t)^{-1}\big] Z_i \\ &+ \big[Z_i, (\lambda - \mathcal{L}_2^t)^{-1}\big] \nabla_{t,e} + \big[\nabla_{t,e}, [Z_i, (\lambda - \mathcal{L}_2^t)^{-1}]\big]. \end{aligned} \tag{3.4.121}$$

Since $\nabla_{t,e}$ is formally self-adjoint with respect to $\big\langle \cdot, \cdot \big\rangle_{t,0}$, by Theorem 3.22 for $s \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,

$$\begin{aligned} \Big\|(\lambda - \mathcal{L}_2^t)^{-1} \nabla_{t,e} Z_i s\Big\|_0 \leqslant& \Big\|(\lambda - \mathcal{L}_2^t)^{-1}\Big\|_t^{-1,0} \cdot \Big\|\nabla_{t,e} Z_i s\Big\|_{t,-1} \\ \leqslant& C(1 + |\lambda|^2) \sup_{s_1 \neq 0} \frac{\Big|\big\langle \nabla_{t,e}(Z_i s), s_1\big\rangle\Big|}{\|s_1\|_{t,1}} \\ \leqslant& C(1 + |\lambda|^2) \big\|Z_i s\big\|_0. \end{aligned} \tag{3.4.122}$$

From Theorem 3.22 and Proposition 3.23,

$$\begin{aligned} \Big\|\big[\nabla_{t,e}, (\lambda - \mathcal{L}_2^t)^{-1}\big] Z_i s\Big\|_0 =& \Big\|(\lambda - \mathcal{L}_2^t)^{-1} \big[\nabla_{t,e}, \mathcal{L}_2^t\big] (\lambda - \mathcal{L}_2^t)^{-1} Z_i s\Big\|_0 \\ \leqslant& C(1 + |\lambda|^2) \Big\|\big[\nabla_{t,e}, \mathcal{L}_2^t\big] (\lambda - \mathcal{L}_2^t)^{-1} Z_i s\Big\|_{t,-1} \\ \leqslant& C(1 + |\lambda|^2) \Big\|(\lambda - \mathcal{L}_2^t)^{-1} Z_i s\Big\|_{t,1} \\ \leqslant& C(1 + |\lambda|^2) \Big\|Z_i s\Big\|_0. \end{aligned} \tag{3.4.123}$$

Similarly,

$$\begin{aligned}\Big\|\big[Z_i,(\lambda-\mathcal{L}_2^t)^{-1}\big]\nabla_{t,e}s\Big\|_0 &\leqslant C(1+|\lambda|^2)^2\cdot\big\|s\big\|_0,\\ \Big\|\big[\nabla_{t,e},[Z_i,(\lambda-\mathcal{L}_2^t)^{-1}]\big]s\Big\|_0 &\leqslant C(1+|\lambda|^2)^3\cdot\big\|s\big\|_0.\end{aligned} \tag{3.4.124}$$

Combining (3.4.121)–(3.4.124) we get

$$\Big\|\nabla_{t,e}Z_i(\lambda-\mathcal{L}_2^t)^{-1}s\Big\|_0 \leqslant C(1+|\lambda|^2)^3\Big(\big\|s\big\|_0+\big\|Z_is\big\|_0\Big). \tag{3.4.125}$$

For $m\in\mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\big\{\nabla_{t,e_{i_1}},\cdots,\nabla_{t,e_{i_j}}\big\}_{j\leqslant m}$. For $k,r\in\mathbb{N}^*$, let

$$I_{k,r}=\Big\{(\mathbf{k},\mathbf{r})=(k_i,r_i),\sum_{i=0}^{j}k_j=k+j,\sum_{i=1}^{j}r_j=r,k_i,r_i\in\mathbb{N}^*\Big\}. \tag{3.4.126}$$

For $(\mathbf{k},\mathbf{r})\in I_{k,r},\lambda\in\delta\cup\gamma,t\in(0,t_0]$, set

$$A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)=(\lambda-\mathcal{L}_2^t)^{-k_0}\frac{\partial^{r_1}}{\partial t^{r_1}}(\lambda-\mathcal{L}_2^t)^{-k_1}\cdots\frac{\partial^{r_j}}{\partial t^{r_j}}(\lambda-\mathcal{L}_2^t)^{-k_j}. \tag{3.4.127}$$

Then there exist $a_{\mathbf{r}}^{\mathbf{k}}\in\mathbb{R}$ such that

$$\frac{\partial^r}{\partial t^r}(\lambda-\mathcal{L}_2^t)^{-k}=\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}}a_{\mathbf{r}}^{\mathbf{k}}A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t). \tag{3.4.128}$$

Theorem 3.26. *For any $m\in\mathbb{N},k>2(m+r+1),(\mathbf{k},\mathbf{r})\in I_{k,r}$, there exist $C>0,N\in\mathbb{N}$ such that for $\lambda\in\delta\cup\gamma,t\in(0,t_0],Q,Q'\in\mathcal{Q}^m$,*

$$\Big\|QA_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q'\Big\|_{t,0}\leqslant C(1+|\lambda|)^N\sum_{|\beta|\leqslant 2r}\big\|Z^\beta s\big\|_{t,0}. \tag{3.4.129}$$

Proof. From Theorem 3.24, we find that for $Q\in\mathcal{Q}^m$, there exist $C_m>0$ and $N\in\mathbb{N}$ such that for any $\lambda\in\delta\cup\gamma$,

$$\Big\|Q(\lambda-\mathcal{L}_2^t)^{-m}\Big\|_t^{0,0}\leqslant C_m(1+|\lambda|^2)^N. \tag{3.4.130}$$

Since the operator $\mathcal{L}_2^t$ is formally self-adjoint, after taking adjoint of (3.4.130) we have for any $\lambda\in\delta\cup\gamma$,

$$\Big\|(\lambda-\mathcal{L}_2^t)^{-m}Q\Big\|_t^{0,0}\leqslant C_m(1+|\lambda|^2)^N. \tag{3.4.131}$$

From (3.4.130) and (3.4.131), we obtain (3.4.129) for $r=0$.

Consider now $r>0$. By (3.4.68), $\frac{\partial^r}{\partial t^r}\mathcal{L}_2^t$ is a combination of

$$\frac{\partial^{r_1}}{\partial t^{r_1}}\big(g^{ij}(tZ)\big)\big(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i}\big)\big(\frac{\partial^{r_3}}{\partial t^{r_3}}\nabla_{t,e_j}\big),\ \frac{\partial^{r_1}}{\partial t^{r_1}}\big(d(tZ)\big),\ \frac{\partial^{r_1}}{\partial t^{r_1}}\big(d_j(tZ)\big)\big(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i}\big), \tag{3.4.132}$$

where $d(Z), d_i(Z)$ and their derivatives in Z are uniformly bounded for $Z \in \mathbb{R}^{2n}$. Now $\frac{\partial^{r_1}}{\partial t^{r_1}}\big(d(tZ)\big)$ (resp. $\big(\frac{\partial^{r_1}}{\partial t^{r_1}}\nabla_{t,e_i}\big)$) ($r_1 \geqslant 1$), are functions of the type $d'(tZ)Z^\beta, |\beta| \leqslant r_1$ (resp. $\leqslant r_1+1$) and $d'(Z)$ and its derivatives in Z are bounded smooth functions of Z.

Let $\mathcal{R}'_t$ be the family of operators of the type

$$\mathcal{R}'_t = \Big\{ \big[f_{j_1}Q_{j_1}, [f_{j_2}Q_{j_2}, \cdots, [f_{j_l}Q_{j_l}, \mathcal{L}^t_2]\cdots]\big] \Big\} \tag{3.4.133}$$

with f_{j_i} smooth bounded (with their derivatives) functions and $Q_{j_i} \in \big\{\nabla_{t,e_l}, Z_l\big\}_{l=1}^{2n}$.

We hand now the operator $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$. We shall move first all the terms $d'(tZ)Z^\beta$ (defined above) to the right-hand side of the operator. To do so, we always use the commutator trick as in the proof of Theorem 3.24, i.e., each time we perform only with the commutation with Z_i or $d'(tZ)Z_i$ (not directly with Z^β for $|\beta|>1$). After this step, $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ takes the form

$$\sum_{|\beta|\leqslant 2r} L^t_\beta Q''_\beta Z^\beta, \tag{3.4.134}$$

where Q''_β is obtained form Q' and its commutations with $d'(tZ)Z^\beta$ (then $Q''_\beta \in \mathcal{Q}^m$), and L^t_β is a linear combination of operators of the form

$$(\lambda-\mathcal{L}^t_2)^{-j_0}R_1\nabla_{t,e_{j_1}}R_2(\lambda-\mathcal{L}^t_2)^{-j_1}R_3\nabla_{t,e_{j_2}}R_4\cdots(\lambda-\mathcal{L}^t_2)^{-j_s} \tag{3.4.135}$$

with R_j of the form

$$\big[f_{j_1}Q_{j_1}, [f_{j_2}Q_{j_2}, \cdots, [f_{j_l}Q_{j_l}, \mathcal{L}^t_2]\cdots]\big],\ Q_{j_i} \in \big\{Z_l\big\}_{l=1}^{2n}. \tag{3.4.136}$$

Now we move all the terms ∇_{t,e_i} (from L^t_β) in (3.4.134) to the left side of Q''_β, then we finally get that $QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is of the form

$$\sum_{|\beta|\leqslant 2r} \mathcal{L}^t_\beta Z^\beta, \tag{3.4.137}$$

where $\mathcal{L}^t_\beta$ is a linear combination of operators of the form

$$Q(\lambda-\mathcal{L}^t_2)^{-k'_0}R'_1(\lambda-\mathcal{L}^t_2)^{-k'_1}R'_2\cdots R'_{l'}(\lambda-\mathcal{L}^t_2)^{-k'_{l'}}Q'''Q''_\beta \tag{3.4.138}$$

with

$$R'_1,\cdots,R'_{l'} \in \mathcal{R}'_t,\ Q''' \in \mathcal{Q}^{2r}. \tag{3.4.139}$$

By (3.4.127) we have

$$\sum_{j=0}^{l'} k'_j \geqslant \sum_{l=0}^{j} k_l + l' = k+j+l' \geqslant k+l'+1. \tag{3.4.140}$$

It is a consequence of (3.4.140) that we can split (3.4.138) into two parts:

$$\begin{aligned} Q_1 :=& Q(\lambda-\mathcal{L}_2^t)^{-k_0'}R_1'(\lambda-\mathcal{L}_2^t)^{-k_1'}R_2'\cdots R_i'(\lambda-\mathcal{L}_2^t)^{-k_i''},\\ Q_2 :=& (\lambda-\mathcal{L}_2^t)^{-(k_i'-k_i'')}R_{i+1}'(\lambda-\mathcal{L}_2^t)^{-k_{i+1}'}R_{i+2}'\cdots R_{l'}'(\lambda-\mathcal{L}_2^t)^{-k_{l'}'}Q'''Q_\beta'' \end{aligned} \tag{3.4.141}$$

with

$$\begin{aligned} &k_0'+k_1'+\cdots+k_{i-1}'+k_i''\geqslant m+i,\\ &k_i'-k_i''+k_{i+1}'+\cdots+k_{l'}'\geqslant m+2r+(l'-i). \end{aligned} \tag{3.4.142}$$

By (3.4.113) we obtain

$$\left\|(\lambda-\mathcal{L}_2^t)^{-1}s\right\|_{t,m+1}\leqslant C(1+|\lambda|^2)^N\left\|s\right\|_{t,m}. \tag{3.4.143}$$

Similarly as the proof of (3.4.113), for $R'\in\mathcal{R}_t'$

$$\left\|(\lambda-\mathcal{L}_2^t)^{-1}R'(\lambda-\mathcal{L}_2^t)^{-1}s\right\|_{t,m+1}\leqslant C(1+|\lambda|^2)^N\left\|s\right\|_{t,m}. \tag{3.4.144}$$

Combining (3.4.113), (3.4.142) and (3.4.144), we get

$$\begin{aligned} \left\|Q_1 s\right\|_0 &= \left\|Q(\lambda-\mathcal{L}_2^t)^{-k_0'}R_1'(\lambda-\mathcal{L}_2^t)^{-k_1'}R_2'\cdots R_i'(\lambda-\mathcal{L}_2^t)^{-k_i''}s\right\|_0\\ &\leqslant \left\|(\lambda-\mathcal{L}_2^t)^{-k_0'}R_1'(\lambda-\mathcal{L}_2^t)^{-k_1'}R_2'\cdots R_i'(\lambda-\mathcal{L}_2^t)^{-k_i''}s\right\|_{t,m}\\ &\leqslant C(1+|\lambda|^2)^{N_0}\Big\|(\lambda-\mathcal{L}_2^t)^{-1}R_1'(\lambda-\mathcal{L}_2^t)^{-1}\times\\ &\qquad\qquad (\lambda-\mathcal{L}_2^t)^{-(k_1'-1)}R_2'\cdots R_i'(\lambda-\mathcal{L}_2^t)^{-k_i''}s\Big\|_{t,m-k_0'+1}\\ &\leqslant C(1+|\lambda|^2)^{N_1}\left\|(\lambda-\mathcal{L}_2^t)^{-(k_1'-1)}R_2'\cdots R_i'(\lambda-\mathcal{L}_2^t)^{-k_i''}s\right\|_{t,m-k_0'}\\ &\leqslant \cdots\cdots\\ &\leqslant C(1+|\lambda|^2)^{N_i}\left\|(\lambda-\mathcal{L}_2^t)^{-(k_i''-1)}s\right\|_{t,m-\sum_{j=0}^{i-1}k_0'+i-1}\\ &\leqslant C(1+|\lambda|^2)^{N}\left\|(\lambda-\mathcal{L}_2^t)^{-(k_i''+\sum_{j=0}^{i-1}k_0'-m-i)}s\right\|_{t,0}\\ &\leqslant C(1+|\lambda|^2)^{N}\left\|s\right\|_{t,0}. \end{aligned}$$

That is

$$\left\|Q_1\right\|_t^{0,0}\leqslant C(1+|\lambda|^2)^N. \tag{3.4.145}$$

Similarly we have

$$\left\|Q_2\right\|_t^{0,0}\leqslant C(1+|\lambda|^2)^N. \tag{3.4.146}$$

The proof of Theorem 3.26 is complete. □

Clearly, as $t \to 0$, the limit of the norm $\|\cdot\|_{t,m}$ exists and we denote it by $\|\cdot\|_{0,m}$. Note $\|\cdot\|_{t,0} = \|\cdot\|_{0,0}$.

Theorem 3.27. *For any $r \geqslant 0, k > 0$, there exist $C > 0, N \in \mathbb{N}$ such that for $t \in [0, t_0], \lambda \in \delta \cup \gamma, s \in C_0^\infty(X_0, \mathbb{E}_{x_0})$,*

$$\Big\|\Big(\frac{\partial^r \mathcal{L}_2^t}{\partial t^r} - \frac{\partial^r \mathcal{L}_2^t}{\partial t^r}\Big|_{t=0}\Big)s\Big\|_{t,-1} \leqslant Ct \sum_{|\alpha|\leqslant r+3} \big\|Z^\alpha s\big\|_{0,1}, \tag{3.4.147}$$

$$\Big\|\Big(\frac{\partial^r}{\partial t^r}(\lambda - \mathcal{L}_2^t)^{-k} - \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0)\Big)s\Big\|_{0,0} \leqslant Ct(1+|\lambda|^2)^N \sum_{|\alpha|\leqslant 4r+3} \big\|Z^\alpha s\big\|_{0,0}.$$

Proof. By the definition of $\|\cdot\|_{t,m}$, for $t \in [0,1], k \geqslant 1$,

$$\big\|s\big\|_{t,k} \leqslant C \sum_{|\alpha|\leqslant k} \big\|Z^\alpha s\big\|_{0,k}, \quad \big\|s\big\|_{0,k} \leqslant C \sum_{|\alpha|\leqslant k} \big\|Z^\alpha s\big\|_{t,k}. \tag{3.4.148}$$

Combining (3.4.148) and the Taylor expansion for (3.4.68), we find that if s_1, s_2 have compact support, then

$$\Big|\Big\langle\Big(\frac{\partial^\alpha \mathcal{L}_2^t}{\partial t^r} - \frac{\partial^\alpha \mathcal{L}_2^t}{\partial t^r}\Big|_{t=0}\Big)s_1, s_2\Big\rangle_{t,0}\Big| \leqslant Ct\big\|s_2\big\|_{t,1} \cdot \sum_{|\alpha|\leqslant r+3} \big\|Z^\alpha s_1\big\|_{0,1}, \tag{3.4.149}$$

where the upper bound $r+3$ of $|\alpha|$ comes from the term

$$-g^{ij}\big(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma_{ij}^k \nabla_{t,e_k}\big). \tag{3.4.150}$$

Now the first inequality in (3.4.147) follows immediately from (3.4.149).

It is a consequence of the first equation of (3.4.68) that

$$\nabla_{t,e_i} = \nabla_{0,e_i} + t g_i(Z) + O(t^2), \tag{3.4.151}$$

where $g_i(Z)$ is a polynomial in Z of degree 2. Substituting (3.4.151) into the second equation of (3.4.68),

$$\mathcal{L}_2^t - \mathcal{L}_2^0 = t h_j \nabla_{e_j} + O(t^2) \tag{3.4.152}$$

with $h_i(Z)$ polynomial in Z of degree 3.

Note

$$(\lambda - \mathcal{L}_2^t)^{-1} = (\lambda - \mathcal{L}_2^0)^{-1} - (\lambda - \mathcal{L}_2^t)^{-1}(\mathcal{L}_2^t - \mathcal{L}_2^0)(\lambda - \mathcal{L}_2^0)^{-1}. \tag{3.4.153}$$

By passing to the limit, we obtain that Theorem 3.22, Proposition 3.23 and Theorem 3.24 still hold for $t = 0$. Using Theorem 3.22 for $t = 0$ and (3.4.153),

$$\begin{aligned} &\Big\|\Big((\lambda - \mathcal{L}_2^t)^{-1} - (\lambda - \mathcal{L}_2^0)^{-1}\Big)s\Big\|_{0,0} \\ =&\Big\|(\lambda - \mathcal{L}_2^t)^{-1}(\mathcal{L}_2^t - \mathcal{L}_2^0)(\lambda - \mathcal{L}_2^0)^{-1}s\Big\|_{0,0} \\ \leqslant& C(1+|\lambda|^2)^N \Big\|(\mathcal{L}_2^t - \mathcal{L}_2^0)(\lambda - \mathcal{L}_2^0)^{-1}s\Big\|_{t,-1}. \end{aligned} \tag{3.4.154}$$

Substituting the first inequality of (3.4.147) for $r=0$ into (3.4.154) yields

$$\left\|\left((\lambda-\mathcal{L}_2^t)^{-1}-(\lambda-\mathcal{L}_2^0)^{-1}\right)s\right\|_{0,0}\leqslant Ct(1+|\lambda|^2)^N\sum_{|\alpha|\leqslant 3}\left\|Z^\alpha(\lambda-\mathcal{L}_2^0)^{-1}s\right\|_{0,1}. \quad (3.4.155)$$

Finally using (3.4.113) for $t=0$, we get

$$\left\|\left((\lambda-\mathcal{L}_2^t)^{-1}-(\lambda-\mathcal{L}_2^0)^{-1}\right)s\right\|_{0,0}\leqslant Ct(1+|\lambda|^2)^N\sum_{|\alpha|\leqslant 3}\left\|Z^\alpha s\right\|_{0,0}. \quad (3.4.156)$$

If we denote $\mathcal{L}_{\lambda,t}=\lambda-\mathcal{L}_2^t$, then

$$A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)-A_{\mathbf{r}}^{\mathbf{k}}(\lambda,0)=\sum_{i=1}^{j}F_i+\sum_{i=0}^{j}G_i \quad (3.4.157)$$

with

$$\begin{aligned}F_i=&\mathcal{L}_{\lambda,t}^{-k_0}\frac{\partial^{r_1}\mathcal{L}_2^t}{\partial t^{r_1}}\mathcal{L}_{\lambda,t}^{-k_1}\cdots\frac{\partial^{r_{i-1}}\mathcal{L}_2^t}{\partial t^{r_{i-1}}}\mathcal{L}_{\lambda,t}^{-k_{i-1}}\cdot\left(\frac{\partial^{r_i}\mathcal{L}_2^t}{\partial t^{r_i}}-\frac{\partial^{r_i}\mathcal{L}_2^t}{\partial t^{r_i}}\Big|_{t=0}\right)\mathcal{L}_{\lambda,0}^{-k_i}\cdot\\&\left(\frac{\partial^{r_{i+1}}\mathcal{L}_2^t}{\partial t^{r_{i+1}}}\Big|_{t=0}\right)\mathcal{L}_{\lambda,0}^{-k_{i+1}}\cdots\left(\frac{\partial^{r_j}\mathcal{L}_2^t}{\partial t^{r_j}}\Big|_{t=0}\right)\mathcal{L}_{\lambda,0}^{-k_j}\end{aligned} \quad (3.4.158)$$

and

$$\begin{aligned}G_i=&\mathcal{L}_{\lambda,t}^{-k_0}\frac{\partial^{r_1}\mathcal{L}_2^t}{\partial t^{r_1}}\mathcal{L}_{\lambda,t}^{-k_1}\cdots\frac{\partial^{r_{i-1}}\mathcal{L}_2^t}{\partial t^{r_{i-1}}}\mathcal{L}_{\lambda,t}^{-k_{i-1}}\cdot\frac{\partial^{r_i}\mathcal{L}_2^t}{\partial t^{r_i}}\left(\mathcal{L}_{\lambda,t}^{-k_i}-\mathcal{L}_{\lambda,0}^{-k_i}\right)\cdot\\&\left(\frac{\partial^{r_{i+1}}\mathcal{L}_2^t}{\partial t^{r_{i+1}}}\Big|_{t=0}\right)\mathcal{L}_{\lambda,0}^{-k_{i+1}}\cdots\left(\frac{\partial^{r_j}\mathcal{L}_2^t}{\partial t^{r_j}}\Big|_{t=0}\right)\mathcal{L}_{\lambda,0}^{-k_j}.\end{aligned} \quad (3.4.159)$$

Similar to the proof of (3.4.113) we have

$$\left\|Z^\alpha(\lambda-\mathcal{L}_2^t)^{-1}s\right\|_{t,0}\leqslant C_{\alpha,m}(1+|\lambda|^2)^N\sum_{\beta\leqslant\alpha}\left\|Z^\beta s\right\|_{t,0}. \quad (3.4.160)$$

Combining the structure of $\frac{\partial^r\mathcal{L}_2^t}{\partial t^r}$ (a combination of (3.4.132)), (3.4.113), (3.4.160) and the first inequality of (3.4.147), we obtain

$$\left\|F_i s\right\|_{0,0}\leqslant Ct(1+|\lambda|^2)^N\sum_{|\alpha|\leqslant 4r+3}\left\|Z^\alpha s\right\|_{0,0}. \quad (3.4.161)$$

Similarly by (3.4.156),

$$\left\|G_i s\right\|_{0,0}\leqslant Ct(1+|\lambda|^2)^N\sum_{|\alpha|\leqslant 4r+3}\left\|Z^\alpha s\right\|_{0,0}. \quad (3.4.162)$$

Now the second inequality of (3.4.147) follows from (3.4.128), (3.4.157), (3.4.161) and (3.4.162). The proof of Theorem 3.27 is complete. □

3.4.4 Uniform estimate on the Bergman kernel

Denote by $\mathcal{P}_{0,t}$ the spectral projection from $\big(L^2(X_0, \mathbb{E}_{x_0}), \|\cdot\|_0\big)$ onto $\mathrm{Ker}(\mathcal{L}_2^t)$. It is a consequence of (3.4.87) that

$$\mathcal{P}_{0,t} = \frac{1}{2\pi i}\int_\delta (\lambda - \mathcal{L}_2^t)^{-1} d\lambda. \tag{3.4.163}$$

Our next step is to convert the estimates for the resolvent $(\lambda - \mathcal{L}_2^t)^{-1}$ into estimates for the spectral projection $\mathcal{P}_{0,t}$ via the formula (3.4.163). Let $\mathcal{P}_{0,t}(Z, Z')$ (with $Z, Z' \in X_0$) be the smooth kernel of $\mathcal{P}_{0,t}$ with respect to $dv_{TX}(Z')$.

Let $\pi : TX \times_X TX \to X$ be the natural projection form the fibrewise product of TX on X. Note that $\mathcal{L}_2^t$ is a family of differential operators on $T_{x_0}X$ with coefficients in $\mathrm{End}(\mathbb{E})_{x_0}$. Thus we can view $\mathcal{P}_{0,t}(Z, Z')$ as a smooth section of $\pi^*\big(\mathrm{End}(\mathbb{E})\big)$ over $TX \times_X TX$ by identifying a section $s \in C^\infty\Big(TX \times_X TX, \pi^*\big(\mathrm{End}(\mathbb{E})\big)\Big)$ with the family $(s_x)_{x\in X}$, where $s_x = s\big|_{\pi^{-1}(x)}$. Let $\nabla^{\mathrm{End}(\mathbb{E})}$ be the connection of $\mathrm{End}(\mathbb{E})$ induced by $\nabla^{\mathbb{E}}$ (which is in turn induced by ∇^B and ∇^E). Then $\nabla^{\pi^*(\mathrm{End}(\mathbb{E}))}$ induces naturally a C^m-norm of s for the parameter $x_0 \in X$. In the rest of this section, we will denote by $C^m(X)$ the C^m-norm for the parameter $x_0 \in X$.

Theorem 3.28. *For any $m, m', r \in \mathbb{N}, a > 0$, there exists $C > 0$ such that for $t \in (0, t_0], Z, Z' \in T_{x_0}X, |Z|, |Z'| \leqslant a$,*

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \Big|\frac{\partial^{|\alpha|+|\alpha|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t}(Z, Z')\Big|_{C^{m'}(X)} \leqslant C. \tag{3.4.164}$$

Proof. It is a consequence of (3.4.163) that

$$\mathcal{P}_{0,t} = \frac{1}{2\pi i}\int_\delta \lambda^{k-1}(\lambda - \mathcal{L}_2^t)^{-k} d\lambda. \tag{3.4.165}$$

From (3.4.130), (3.4.131) and (3.4.165) we obtain

$$\big\|Q\mathcal{P}_{0,t}Q'\big\|_t^{0,0} \leqslant C_m, \quad \text{for } Q, Q' \in \mathcal{Q}^m. \tag{3.4.166}$$

Let $\big|\cdot\big|_{(a),m}$ be the usual Sobolev norm on $C^\infty(B_{a+1}, \mathbb{E}_{x_0})$ induced by $h^{\mathbb{E}_{x_0}}$ and the volume form $dv_{TX}(Z)$, i.e., for $s \in C^\infty(B_{a+1}, \mathbb{E}_{x_0})$,

$$\big|s\big|_{(a),0}^2 = \int_{B_{a+1}} \big|s(Z)\big|^2 dv_{TX}(Z),$$
$$\big|s\big|_{(a),m}^2 = \sum_{l=0}^m \sum_{i_1,\cdots,i_l=1}^{2n} \big\|\nabla_{e_{i_1}}\cdots\nabla_{e_{i_l}} s\big\|_{(a),0}. \tag{3.4.167}$$

From (3.4.68), (3.4.79) and (3.4.167), we get that for $m > 0$, there exists $C_m > 0$ such that for $s \in C^\infty(X_0, \mathbb{E}_{x_0}), \mathrm{supp}(s) \subset B_{a+1}$,

$$\frac{C_m}{(1+a)^m}\big\|s\big\|_{t,m} \leqslant \big|s\big|_{(a),m} \leqslant C_m(1+a)^m\big\|s\big\|_{t,m}. \tag{3.4.168}$$

If $Q'' \in \mathcal{Q}^m$ and Q'' has compact support in B_{a+1}, then for every $s \in L^2(X_0, \mathbb{E}_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $|Z| \leqslant a$,

$$\begin{aligned}\big\langle Q''_{\bullet} Q_Z \mathcal{P}_{0,t}(Z,\cdot), s(\cdot)\big\rangle_{t,0} =& \int_{\mathbb{R}^{2n}} Q_Z \mathcal{P}_{0,t}(Z,Z')(Q''s)(Z')dv_{TX}(Z') \\ =& \big(QP_{0,t}Q''s\big)(Z).\end{aligned} \tag{3.4.169}$$

Using (3.4.168), (3.4.169) and Sobolev inequalities, we obtain

$$\begin{aligned}\Big\|Q''Q_ZP_{0,t}(Z,\cdot)\Big\|_{t,0} =& \sup_{\|s\|_0=1} \Big|\big\langle Q''_{\bullet} Q_Z \mathcal{P}_{0,t}(Z,\cdot), s(\cdot)\big\rangle_{t,0}\Big| \\ =& \sup_{\|s\|_0=1} \Big|\big(QP_{0,t}Q''s\big)(Z)\Big| \\ \leqslant& C \sup_{\|s\|_0=1} \Big|QP_{0,t}Q''s\Big|_{(a),n+1} \\ \leqslant& C(1+a)^{n+1} \sup_{\|s\|_0=1} \Big|QP_{0,t}Q''s\Big|_{t,n+1}.\end{aligned} \tag{3.4.170}$$

Substituting (3.4.166) into (3.4.170), we get

$$\Big\|Q''Q_ZP_{0,t}(Z,\cdot)\Big\|_{t,0} \leqslant C(1+a)^{n+1}. \tag{3.4.171}$$

Combining (3.4.168), (3.4.171) and Sobolev inequalities, we have

$$\begin{aligned}\sup_{|Z|,|Z'|\leqslant a} \Big|Q'_{Z'}Q_Z\mathcal{P}_{0,t}(Z,Z')\Big| \leqslant& C \sup_{|Z|\leqslant a} \Big|Q'_{\bullet}Q_Z\mathcal{P}_{0,t}(Z,\cdot)\Big|_{(a),n+1} \\ \leqslant& C(1+a)^{n+1} \sup_{|Z|\leqslant a} \Big|Q'_{\bullet}Q_Z\mathcal{P}_{0,t}(Z,\cdot)\Big|_{t,n+1} \\ \leqslant& C(1+a)^{2n+2}.\end{aligned} \tag{3.4.172}$$

From (3.4.68) and (3.4.172) we obtain the estimates (3.4.164) for $r = m' = 0$.

To obtain (3.4.164) for $r \geqslant 1$ and $m' = 0$, note that

$$\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t} = \frac{1}{2\pi i}\int_{\delta} \lambda^{k-1} \frac{\partial^r}{\partial t^r}(\lambda - \mathcal{L}_2^t)^{-k} d\lambda. \tag{3.4.173}$$

By (3.4.128) and (3.4.129) we know

$$\Big\|Q\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t}Q'\Big\|_t^{0,0} \leqslant C_m, \quad \text{for } Q, Q' \in \mathcal{Q}^m. \tag{3.4.174}$$

Then by the above argument, we get the estimate (3.4.164) for $r \geqslant 1$ and $m' = 0$.

Finally for $U \in TX$,

$$\nabla_U^{\pi^*(\mathrm{End}(\mathbb{E}))}\mathcal{P}_{0,t} = \frac{1}{2\pi i}\int_{\delta} \lambda^{k-1} \nabla_U^{\pi^*(\mathrm{End}(\mathbb{E}))}(\lambda - \mathcal{L}_2^t)^{-k} d\lambda. \tag{3.4.175}$$

Clearly,

$$\nabla_U^{\pi^*(\mathrm{End}(\mathbb{E}))}(\lambda - \mathcal{L}_2^t)^{-k} = \sum_{i=1}^{k} a_i A_i, \tag{3.4.176}$$

where $a_i \in \mathbb{R}$ and

$$A_i = (\lambda - \mathcal{L}_2^t)^{-i}\Big(\nabla_U^{\pi^*(\mathrm{End}(\mathbb{E}))}\mathcal{L}_2^t\Big)(\lambda - \mathcal{L}_2^t)^{-(k+1-i)}. \tag{3.4.177}$$

Since $\nabla_U^{\pi^*(\mathrm{End}(\mathbb{E}))}\mathcal{L}_2^t$ is a differential operator on $T_{x_0}X$ with the same structure on $\mathcal{L}_2^t$, i.e., it has the same type as (3.4.110), we know from the proof of (3.4.129),

$$\Big\|Q\big(\nabla_U^{\pi^*(\mathrm{End}(\mathbb{E}))}(\lambda - \mathcal{L}_2^t)^{-k}\big)Q'\Big\|_t^{0,0} \leqslant C_m, \quad \text{for } Q, Q' \in \mathcal{Q}^m. \tag{3.4.178}$$

Then using the above argument, we conclude that (3.4.164) holds for $m' \geqslant 1$. The proof of Theorem 3.28 is complete. □

For k large enough, set

$$\begin{aligned} \mathcal{F}_r =& \frac{1}{2\pi i \cdot r!}\int_\delta \lambda^{k-1} \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0)d\lambda,\\ \mathcal{F}_{r,t} =& \frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t} - \mathcal{F}_r. \end{aligned} \tag{3.4.179}$$

Let $\mathcal{F}_r(Z,Z')$ $(Z,Z' \in T_{x_0}X)$ be the smooth kernel of $\mathcal{F}_r$ with respect to $dv_{TX}(Z')$. Then

$$\mathcal{F}_r(Z,Z') \in C^\infty\big(TX \times_X TX, \pi^*(\mathrm{End}(\mathbb{E}))\big). \tag{3.4.180}$$

Theorem 3.29. *For $a > 0$, there exists $C > 0$ such that for $t \in (0,1], Z, Z' \in T_{x_0}X$, and $|Z|, |Z'| \leqslant a$,*

$$\Big|\mathcal{F}_r(Z,Z')\Big| \leqslant Ct^{\frac{1}{2n+1}}. \tag{3.4.181}$$

Proof. Combining (3.4.173) and (3.4.179),

$$\mathcal{F}_{r,t} = \frac{1}{2\pi i \cdot r!}\int_\delta \lambda^{k-1}\Big(\frac{\partial^r}{\partial t^r}(\lambda - \mathcal{L}_2^t)^{-k} - \frac{\partial^r}{\partial t^r}(\lambda - \mathcal{L}_2^t)^{-k}\Big|_{t=0}\Big)d\lambda. \tag{3.4.182}$$

It is a consequence of (3.4.147) that

$$\big\|\mathcal{F}_{r,t}\big\|_{(a),0} \leqslant Ct. \tag{3.4.183}$$

Let $\sigma : \mathbb{R}^{2n} \to [0,1]$ be a smooth function with compact support in B_1, equal 1 near 0. Take $\varsigma \in (0,1]$. By the proof of Theorem 3.28, $\mathcal{F}_r$ (hence $\mathcal{F}_{r,t}$) verifies the similar inequalities as in (3.4.164), i.e.,

$$\sup_{|\alpha|+|\alpha'|\leqslant m}\Big|\frac{\partial^{|\alpha|+|\alpha|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\mathcal{F}_r(Z,Z')\Big|_{C^{m'}(X)} \leqslant C. \tag{3.4.184}$$

For $|Z|, |Z'| \leqslant a, U_1, U_2 \in \mathbb{E}_{x_0}$, set

$$\mathcal{K} := \int_{T_{x_0}X\times T_{x_0}X} \Big\langle \mathcal{F}_{r,t}(Z-W, Z'-W')U_1, U_2\Big\rangle \frac{1}{\varsigma^{4n}}\sigma(\frac{W}{\sigma})\sigma(\frac{W'}{\varsigma})dv_{TX}(W)dv_{TX}(W').$$

Combining (3.4.164) and (3.4.184) we find

$$\Big|\Big\langle \mathcal{F}_{r,r}(Z, Z')U_1, U_2\Big\rangle - \mathcal{K}\Big| \leqslant C\varsigma|U_1|\cdot|U_2|. \tag{3.4.185}$$

Set

$$s_i(W) = \frac{1}{\varsigma^{2n}}\sigma(\frac{W}{\varsigma})U_i, \text{ for } i = 1, 2;$$
$$\tilde{s}_1(W) = s_1(Z-W),\ \tilde{s}_2(W) = s_2(Z'-W).$$

Then

$$\text{supp}(s_i) \subset B_1 \text{ and } \text{supp}(\tilde{s}_i) \subset B_{a+1}, \text{ for } i = 1, 2, \tag{3.4.186}$$

and

$$\big\|\tilde{s}_i\big\|_0^2 = \big\|s_i\big\|_0^2 = \frac{1}{\varsigma^{2n}}\int_{\mathbb{R}^{2n}} |U_i|^2, \text{ for } i = 1, 2. \tag{3.4.187}$$

It is clear that

$$\begin{aligned}\mathcal{K} :=& \int_{T_{x_0}X\times T_{x_0}X} \Big\langle \mathcal{F}_{r,t}(Z-W, Z'-W')s_1(W'), s_2(W')\Big\rangle dv_{TX}(W)dv_{TX}(W')\\ =& \int_{T_{x_0}X\times T_{x_0}X} \Big\langle \mathcal{F}_{r,t}(W, W')\tilde{s}_1(W'), \tilde{s}_2(W')\Big\rangle dv_{TX}(W)dv_{TX}(W')\\ =&\Big\langle \mathcal{F}_{r,t}\tilde{s}_1, \tilde{s}_2\Big\rangle_{t,0}.\end{aligned} \tag{3.4.188}$$

From (3.4.183), (3.4.187) and (3.4.188) we obtain

$$\big|\mathcal{K}\big| \leqslant \big\|\mathcal{F}_{r,t}\big\|_{(a),0}\cdot\big\|\tilde{s}_1\big\|_0\cdot\big\|\tilde{s}_2\big\|_0 \leqslant \frac{Ct}{\varsigma^{2n}}|U_1|\cdot|U_2|. \tag{3.4.189}$$

Combining (3.4.185) and (3.4.189) we get

$$\Big|\Big\langle \mathcal{F}_{r,r}(Z, Z')U_1, U_2\Big\rangle\Big| \leqslant C(\varsigma + \frac{t}{\varsigma^{2n}})|U_1|\cdot|U_2|. \tag{3.4.190}$$

Set $\varsigma = t^{\frac{1}{2n+1}}$, then

$$\Big|\Big\langle \mathcal{F}_{r,r}(Z, Z')U_1, U_2\Big\rangle\Big| \leqslant Ct^{\frac{1}{2n+1}}|U_1|\cdot|U_2|, \tag{3.4.191}$$

which implies (3.4.181) immediately. The proof of Theorem 3.29 is complete. □

Finally we obtain the following near diagonal estimate for the kernel of $\mathcal{P}_{0,t}$.

Theorem 3.30. *For any $k, m, m' \in \mathbb{N}, a > 0$, there exists $C > 0$ such that for $t \in (0, t_0], Z, Z' \in T_{x_0}X$ and $|Z|, |Z'| \leqslant a$,*

$$\sum_{|\alpha|+|\alpha'|\leqslant m} \Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}}\big(\mathcal{P}_{0,t} - \sum_{r=0}^{k} \mathcal{F}_r t^r\big)(Z, Z')\Big|_{C^{m'}(X)} \leqslant ct^{k+1}. \tag{3.4.192}$$

Proof. By (3.4.179) and (3.4.181) we have

$$\frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t}\Big|_{t=0} = \mathcal{F}_r. \tag{3.4.193}$$

From (3.4.164), (3.4.193) and the Taylor expansion

$$G(t) - \sum_{r=0}^{k} \frac{1}{r!}\frac{\partial^r G}{\partial t^r}(0)t^r = \frac{1}{k!}\int_0^t (t-s)^k \frac{\partial^{k+1}G}{\partial t^{k+1}}(s)ds, \tag{3.4.194}$$

we obtain (3.4.192). This completes the proof of Theorem 3.30. □

3.4.5 Bergman kernel of $\mathcal{L}_2^0$

Recall that $\mathcal{L}_2^0$ is given by (3.4.64). Now we discuss the eigenvalues and eigenfunctions of $\mathcal{L}_2^0$ in detail.

Let $\{v_1, \ldots, v_n\}$ denote an orthonormal basis of $T_{x_0}^{(1,0)}X$ such that (3.1.13) holds. Set

$$e_{2j-1} = \frac{v_j + \overline{v}_j}{\sqrt{2}}, \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(v_j - \overline{v}_j). \tag{3.4.195}$$

Then $\{e_1, e_2, \ldots, e_{2n-1}, e_{2n}\}$ forms an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ induced by $\{e_1, \ldots, e_{2n}\}$ as in (3.4.38).

We also introduce the complex coordinates $z = (z_1, \cdots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ such that $v_j = \sqrt{2}\frac{\partial}{\partial z_j}$ holds at the point $x = x_0$. We identify z to $\sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ and $\overline{z}$ to $\sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j}$ when we consider z and $\overline{z}$ as vector fields. Then $Z = z + \overline{z}$, and $v_j = \sqrt{2}\frac{\partial}{\partial z_j}, \overline{v}_j = \sqrt{2}\frac{\partial}{\partial \overline{z}_j}$. Remark that

$$\Big|\frac{\partial}{\partial z_j}\Big|^2 = \Big|\frac{\partial}{\partial \overline{z}_j}\Big|^2 = \frac{1}{2}, \quad \text{and } |z|^2 = |\overline{z}|^2 = \frac{1}{2}\big|Z\big|^2. \tag{3.4.196}$$

Set

$$\xi = (\overline{z}_1, \cdots, \overline{z}_q, z_{q+1}, \cdots, z_n), \quad \overline{\xi} = (z_1, \cdots, z_q, \overline{z}_{q+1}, \cdots, \overline{z}_n). \tag{3.4.197}$$

It is a consequence of (3.1.13) that

$$\mathbf{J}\frac{\partial}{\partial \xi_j} = \sqrt{-1}\frac{\partial}{\partial \xi_j}, \quad \mathbf{J}\frac{\partial}{\partial \overline{\xi}_j} = -\sqrt{-1}\frac{\partial}{\partial \overline{\xi}_j}, \quad \text{for } j = 1, \cdots, n. \tag{3.4.198}$$

We also identify ξ to $\sum_{j=1}^n \xi_j \frac{\partial}{\partial \xi_j}$ and $\overline{\xi}$ to $\sum_{j=1}^n \overline{\xi}_j \frac{\partial}{\partial \overline{\xi}_j}$ when we consider ξ and $\overline{\xi}$ as vector fields. Then $\xi + \overline{\xi} = Z = z + \overline{z}$. Remark that

$$\Big|\frac{\partial}{\partial \xi_j}\Big|^2 = \Big|\frac{\partial}{\partial \overline{\xi}_j}\Big|^2 = \frac{1}{2}, \quad \text{and } |\xi|^2 = |\overline{\xi}|^2 = \frac{1}{2}\big|Z\big|^2. \tag{3.4.199}$$

Set $u_j = \sqrt{2}\frac{\partial}{\partial \xi_j}$ and

$$f_{2j-1} = \frac{1}{\sqrt{2}}(u_j + \overline{u}_j), \quad f_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(u_j - \overline{u}_j), \quad j = 1, \cdots, n. \tag{3.4.200}$$

Then $\{f_1, \ldots, f_{2n}\}$ is also an orthonormal basis of $T_{x_0}X$.

Set

$$\mathcal{L}_0 = -(\nabla_{0,e_i})^2 - 2n\pi. \tag{3.4.201}$$

Then

$$\mathcal{L}_2^0 = \mathcal{L}_0 - 2\omega_{d,x_0}. \tag{3.4.202}$$

It is very useful to rewrite $\mathcal{L}_0$ in (3.4.201) by using the creation and annihilation operators. Set

$$b_j = -2\nabla_{0,\frac{\partial}{\partial \xi_j}} = -2\frac{\partial}{\partial \xi_j} + \pi\overline{\xi}_j, \quad b_j^+ = 2\nabla_{0,\frac{\partial}{\partial \overline{\xi}_j}} = 2\frac{\partial}{\partial \overline{\xi}_j} + \pi\xi_j, \quad b = (b_1, \cdots b_n). \tag{3.4.203}$$

Then for any polynomial $g(\xi, \overline{\xi})$ on ξ and $\overline{\xi}$,

$$\begin{aligned} [b_i, b_j^+] =& b_i b_j^+ - b_j^+ b_i = -4\pi\delta_{ij}, \quad [b_i, b_j] = [b_i^+, b_j^+] = 0, \\ [g(\xi, \overline{\xi}), b_j] =& 2\frac{\partial}{\partial \xi_j} g(\xi, \overline{\xi}), \qquad [g(\xi, \overline{\xi}), b_j^+] = -2\frac{\partial}{\partial \overline{\xi}_j} g(\xi, \overline{\xi}). \end{aligned} \tag{3.4.204}$$

By (3.4.201), (3.4.203) and (3.4.204), we obtain

$$\mathcal{L}_0 = \sum_{j=1}^n b_j b_j^+. \tag{3.4.205}$$

The following result is due to Ma and Marinescu, see [30, Theorem 8.2.3].

Theorem 3.31. *The spectrum of the restriction of $\mathcal{L}_0$ to $L^2(\mathbb{R}^{2n})$ is given by*

$$\mathrm{Spec}(\mathcal{L}_0|_{L^2(\mathbb{R}^{2n})}) = \big\{4\pi \sum_{j=1}^n \alpha_j \big| \ \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n\big\} \tag{3.4.206}$$

and an orthogonal basis of the eigenspace of $4\pi \sum_{j=1}^n \alpha_j$ is given by

$$b^\alpha\big(\xi^\beta \exp\big(-\frac{\pi}{2}\sum_{j=1}^n |\xi_j|^2\big)\big), \quad \textit{with } \beta \in \mathbb{N}^n. \tag{3.4.207}$$

It is a consequence of (3.4.207) that an orthonormal basis of $\mathrm{Ker}(\mathcal{L}_0)$ is

$$\Big(\frac{\pi^{|\beta|}}{\beta!}\Big)^{\frac{1}{2}} z^\beta \exp\Big(-\frac{\pi}{2}\sum_{j=1}^n |\xi_j|^2\Big)\Big), \quad with\ \beta\in\mathbb{N}^n. \tag{3.4.208}$$

Let $\mathcal{P}$ denote the orthogonal projection from $\big(L^2(\mathbb{R}^{2n}), \|\cdot\|_0\big)$ onto $\mathrm{Ker}(\mathcal{L}_0)$. Let $\mathcal{P}(Z,Z')$ be the smooth kernel of $\mathcal{P}$ with respect to $dv_{TX}(Z')$. From (3.4.208), we get

$$\mathcal{P}(Z,Z') = \exp\Big[-\frac{\pi}{2}\sum_{j=1}^n \big(|\xi_j|^2+|\xi'_j|^2\big) + \pi\sum_{j=1}^n \xi_j\overline{\xi}'_j\Big] \tag{3.4.209}$$

and

$$b_j^+\mathcal{P}=0, \quad (b_j\mathcal{P})(Z,Z') = 2\pi(\overline{\xi}_j-\overline{\xi}'_j)\mathcal{P}(Z,Z'), \quad \text{for } j=1,\cdots,n. \tag{3.4.210}$$

Let P^N be the orthogonal projection from $\big(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0}), \|\cdot\|_{0,0}\big)$ onto $N:=\mathrm{Ker}(\mathcal{L}_2^0)$, and let $P^N(Z,Z')$ be its smooth kernel with respect to $dv_{TX}(Z')$. Denote by $N^\perp$ the orthogonal space of N in $L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$. Set $\mathcal{P}^\perp=\mathrm{Id}-\mathcal{P}$, $P^{N^\perp}=\mathrm{Id}-P^N$. Since

$$w_d\big|_{(\det(\overline{W}^*))^\perp} \leqslant -2\pi, \tag{3.4.211}$$

it is a consequence of (3.4.202) that

$$P^N(Z,Z') = \mathcal{P}(Z,Z') I_{\det(\overline{W}^*)\otimes E}. \tag{3.4.212}$$

3.4.6 Proof of Theorem 3.2

Let $f(\lambda,t)$ be a formal power series on t with values in $\mathrm{End}\big(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})\big)$,

$$f(\lambda,t) = \sum_{r=0} t^r f_r(t), \ f_r(\lambda)\in \mathrm{End}\big(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})\big). \tag{3.4.213}$$

Consider the equation of formal power series for $\lambda\in\delta\cup\gamma$,

$$(\lambda-\mathcal{L}_2^0-\sum_{r=1}^\infty t^r\mathcal{O}_r) f(\lambda,t) = \mathrm{Id}_{L^2(\mathbb{R}^{2n},\ \mathbb{E}_{x_0})}. \tag{3.4.214}$$

We decompose $f(\lambda,t)$ according to the splitting $L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0}) = N\oplus N^\perp$,

$$g_r(\lambda) = P^N f_r(\lambda), \quad f_r^\perp(\lambda) = P^{N^\perp} f_r(\lambda). \tag{3.4.215}$$

Using (3.4.215) and identifying the powers of t in (3.4.214), we get

$$\begin{aligned} g_0(\lambda) =& \frac{1}{\lambda}P^N, \quad f_0^\perp(\lambda) = (\lambda-\mathcal{L}_2^0)^{-1}P^{N^\perp}, \\ f_r^\perp(\lambda) =& (\lambda-\mathcal{L}_2^0)^{-1}\sum_{j=1}^r P^{N^\perp}\mathcal{O}_j f_{r-j}(\lambda), \\ g_r(\lambda) =& \frac{1}{\lambda}\sum_{j=1}^r P^N\mathcal{O}_j f_{r-j}(\lambda). \end{aligned} \tag{3.4.216}$$

Theorem 3.32. *There exist $J_r(Z,Z') \in \mathrm{End}(\mathbb{E})_{x_0}$ polynomials in Z, Z' with the same parity as r and $\deg J_r(Z,Z') \leqslant 3r$, whose coefficients are polynomials in R^{TX}, R^B, R^E (and R^L) and their derivatives of order $\leqslant r-2$ (resp. $\leqslant r$), such that*

$$\mathcal{F}_r(Z,Z') = J_r(Z,Z')\mathcal{P}(Z,Z'),\ \ J_0(Z,Z') = I_{\det(\overline{W}^*)\otimes E}. \tag{3.4.217}$$

Proof. By (3.4.173) and (3.4.193),

$$\mathcal{F}_r = \frac{1}{2\pi i r!}\int_\delta \frac{d^r}{dt^r}(\lambda - \mathcal{L}_2^t)^{-1}\Big|_{t=0} d\lambda = \frac{1}{2\pi i r!}\int_\delta (g_r(\lambda) + f_r^\perp(\lambda))d\lambda. \tag{3.4.218}$$

From Theorem 3.31 and (3.4.202), the only eigenvalues of $\mathcal{L}_2^0$ inside δ is 0. By (3.4.212), (3.4.216) and (3.4.218), we get

$$\mathcal{F}_0 = P^N = \mathcal{P} I_{\det(\overline{W}^*)\otimes E} \tag{3.4.219}$$

and

$$\mathcal{F}_1 = -P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp} - (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N. \tag{3.4.220}$$

The two summands on the righthand side of (3.4.220) are self-adjoint to each other.
By (3.4.209) and (3.4.212),

$$P^N(Z,Z') = e^{-\frac{\pi}{2}\sum_{j=1}^n \left(|\xi_j|^2 + |\xi_j'|^2 - 2\xi_j\overline{\xi}_j'\right)} I_{\det(\overline{W}^*)\otimes E}. \tag{3.4.221}$$

By Theorem 3.19,

$$\mathcal{O}_1 = B_{i,1}\nabla_{e_i} + C_1, \tag{3.4.222}$$

where $B_{i,1}, C_1$ are polynomial in Z and satisfy the following conditions: the coefficients of $B_{i,1}, C_1$ are polynomials of $R^L_{x_0}$ with $\deg_Z B_{i,1} \leqslant 2$, and $\deg_Z C_1 \leqslant 3$.
Substituting (3.4.221) and (3.4.222) into (3.4.220) we get

$$\mathcal{F}_1(Z,Z') = J_1(Z,Z')\mathcal{P}(Z,Z') \tag{3.4.223}$$

with $J_1(Z,Z')$ satisfying $\deg_Z J_1(Z,Z') + \deg_{Z'} J_1(Z,Z')$ is odd, while $\deg_Z J_1(Z,Z') + \deg_{Z'} J_1(Z,Z') \leqslant 3$, and the coefficients of $J_1(Z,Z')$ being polynomials of $R^L_{x_0}$. □

Combining (3.4.216) and (3.4.218), we get the following formula for $\mathcal{F}_2$,

$$\begin{aligned}\mathcal{F}_2 =& (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N - (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N\\ &+ P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp} - P^N\mathcal{O}_2(\mathcal{L}_2^0)^{-1}P^{N^\perp}\\ &+ (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp} - P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-2}P^{N^\perp}\mathcal{O}_1P^N\\ &- P^N\mathcal{O}_1P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-2}P^{N^\perp} - P^{N^\perp}(\mathcal{L}_2^0)^{-2}\mathcal{O}_1P^N\mathcal{O}_1P^N.\end{aligned} \tag{3.4.224}$$

The following near diagonal expansion of the Bergman kernel is the main result of this Section.

Theorem 3.33. *For any $k, m, m' \in \mathbb{N}, a > 0$, there exists $C > 0$ such that for $a > 1$, $Z, Z' \in T_{x_0}X$ and $|Z|, |Z'| \leqslant \frac{a}{\sqrt{p}}$,*

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}} \Big(\frac{1}{p} P_p(Z, Z') - \sum_{r=0}^{r} \mathcal{F}_r(\sqrt{p}Z, \sqrt{p}Z') k^{-\frac{1}{2}}(Z) k^{-\frac{1}{2}}(Z') p^{-\frac{r}{2}} \Big) \Big|_{C^{m'}(X)} \leqslant C p^{-\frac{k-m+1}{2}}. \quad (3.4.225)$$

Proof. It is a consequence of (3.4.86) and (3.4.87) that

$$P_{0,p} = \frac{1}{2\pi i} \int_{\delta'} \left(\lambda - (D_p^{c,A_0})^2\right)^{-1} d\lambda, \quad (3.4.226)$$

where $\delta' = \left\{ z \in \mathbb{C}, |z| = \frac{\pi}{2} p \right\}$.

Combining (3.4.163) and (3.4.226) we get for $Z, Z' \in \mathbb{R}^{2n}$,

$$P_{0,p}(Z, Z') = t^{2n} k^{-\frac{1}{2}}(Z) \mathcal{P}_{0,t}(\frac{Z}{t}, \frac{Z'}{t}) k^{-\frac{1}{2}}(Z'). \quad (3.4.227)$$

From Proposition 3.18, Theorem 3.30 and (3.4.227), we get (3.4.225). □

Proof of Theorem 3.2. Set $Z = Z' = 0$ in (3.4.225). Since Theorem 3.32 implies that $\mathcal{F}_{2r+1}(0, 0) = 0$ we obtain (3.1.11) and

$$\mathbf{b}_r(x_0) = \mathcal{F}_{2r}(0, 0). \quad (3.4.228)$$

Combining (3.4.209), (3.4.219) and (3.4.228), we obtain (3.1.10). □

3.5 A simplified formula for the coefficient b_1

In this Section, we prove the last two terms in the formula (3.4.224) vanish. In subsection 3.5.1, we calculate the second and third terms, i.e., $\mathcal{O}_1, \mathcal{O}_2$, of the Taylor expansion of the rescaled operator $\mathcal{L}_2^t$. In subsection 3.5.2, we establish that the last two terms in (3.4.224) vanish. Then we obtain a simplified formula (3.5.28) for the second coefficient $\mathbf{b}_1$.

3.5.1 The second and third terms in the Taylor expansion of $\mathcal{L}_2^t$

We use freely the notations from Section 3.4. In particular, the operator $\mathcal{L}_2^0$ is given by (3.4.64).

It is an immediately consequence of (3.4.39) that all objects on X_0 with subscript coincides with the original date on X in $B_{2\varepsilon}$, e.g., the connections $\nabla^{L_0}, \nabla^{TX_0}, \nabla^{E_0}$ coincides in $B_{2\varepsilon}$ with ∇^L, ∇^{TX} and ∇^E, respectively. Hence, the cutoff function ρ has no contribution to our calculation of the local date $\mathbf{b}_1(x_0)$. In this sense, we forget it in our subsequent calculations of $\mathbf{b}_1(x_0)$. Then the rescaling (3.4.61) is simplified as follows. For $s \in C^\infty(B_\varepsilon, \mathbb{E}_{x_0})$ and $Z \in B_\varepsilon$, for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}(\delta_t s)(Z) = s(Z/t),\ \nabla_t = \delta_t^{-1} t k^{\frac{1}{2}} \nabla^{B,\Lambda^{0,\bullet}\otimes L^p \otimes E} k^{-\frac{1}{2}} \delta_t,\\ \mathcal{L}_2^t = \delta_t^{-1} t^2 k^{\frac{1}{2}} D_p^2 k^{-\frac{1}{2}} \delta_t.\end{aligned} \tag{3.5.1}$$

If $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ is a multi-index, set $Z^\alpha = Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. Let $(\partial^\alpha R^L)_{x_0}$ be the tensor $(\partial^\alpha R^L)_{x_0}(e_i, e_j) = \partial^\alpha \big(R^L(e_i, e_j)\big)_{x_0}$. We adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$, and most of the time, we will omit the subscript x_0. Let $\mathcal{O}_1', \mathcal{O}_2'$ be the operators defined as [29, (2.5)], [31, (1.30)]:

$$\begin{aligned}\mathcal{O}_1'(Z) =& -\frac{2}{3}(\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j \nabla_{0,e_i} - \frac{1}{3}(\partial_i R^L)_{x_0}(\mathcal{R}, e_i),\\ \mathcal{O}_2'(Z) =& \frac{1}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\Big\rangle \nabla_{0,e_i}\nabla_{0,e_j}\\ &+ \Big[\frac{2}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_j, e_i\Big\rangle - \Big(\frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!} + R^E_{x_0}\Big)(\mathcal{R}, e_i)\Big]\nabla_{0,e_i}\\ &- \frac{1}{4}\nabla_{e_i}\Big(\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R}, e_i)\Big) - \frac{1}{9}\sum_i\Big[\sum_j(\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j\Big]^2\\ &- \frac{1}{12}\Big[\mathcal{L}_0, \Big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\Big\rangle_{x_0}\Big].\end{aligned} \tag{3.5.2}$$

Set

$$\Psi = \frac{1}{4}\, {}^c(dT_{as})_{x_0}. \tag{3.5.3}$$

The following result, an analogue of [29, Theorem 2.2], provides us the explicit expressions of $\mathcal{O}_1$ and $\mathcal{O}_2$.

Theorem 3.34. *There are second order differential operators $\mathcal{O}_r (r \geqslant 1)$ which are self-adjoint with respect to $\|\cdot\|_{0,0}$ on $C_0^\infty(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, and*

$$\begin{aligned}\mathcal{O}_1 =& \mathcal{O}_1' - \pi\sqrt{-1}\Big\langle\big(\nabla^B_{\mathcal{R}}\mathbf{J}\big)e_i, e_j\Big\rangle c(e_i)c(e_j),\\ \mathcal{O}_2 =& \mathcal{O}_2' - R^{B,\Lambda^{0,\bullet}}_{x_0}(\mathcal{R}, e_i)\nabla_{0,e_i} - \frac{\pi}{2}\sqrt{-1}\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\mathcal{R},\mathcal{R}),x_0}e_i, e_j\Big\rangle c(e_i)c(e_j)\\ &+ \frac{1}{2}\Big(R^E_{x_0} + \frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}_{x_0}\big]\Big)(e_i, e_j)c(e_i)c(e_j) + \frac{1}{4}r^X_{x_0} - \Psi\end{aligned} \tag{3.5.4}$$

such that

$$\mathcal{L}_2^t = \mathcal{L}_2^0 + \sum_{r=1}^\infty \mathcal{O}_r t^r. \tag{3.5.5}$$

Proof. We still give the proof for the reader's convenience. From the Lichnerowicz formula (3.2.12) we find that

$$\mathcal{L}_2^t = {}^c(R^L_{tZ}) + t^2\Phi_{tZ} + \mathcal{L}^t, \tag{3.5.6}$$

where

$$\mathcal{L}^t = \delta_t^{-1} k^{\frac{1}{2}} t^2 \Delta^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E} k^{-\frac{1}{2}} \delta_t = -g^{ij}(tZ)\big[\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij,tZ}\nabla_{t,e_k}\big]. \tag{3.5.7}$$

From [29, (2.14)], the Taylor formula of ${}^c(R^L_{tZ})$ is

$$\begin{aligned} {}^c(R^L_{tZ}) =& \frac{1}{2}R^L_{x_0}(e_i,e_j)c(e_i)c(e_j) - \sqrt{-1}\pi t\Big\langle \big(\nabla^B_{\mathcal{R}}\mathbf{J}\big)e_i, e_j\Big\rangle c(e_i)c(e_j) \\ &- \frac{\sqrt{-1}}{2}\pi t^2 \Big\langle \big(\nabla^B\nabla^B\mathbf{J}\big)_{(\mathcal{R},\mathcal{R}),x_0} e_i, e_j\Big\rangle c(e_i)c(e_j) + O(t^3). \end{aligned} \tag{3.5.8}$$

It is a consequence of (3.2.13) that

$$t^2\Phi_{tZ} = t^2\Big\{\frac{r^X_{x_0}}{4} + {}^c\big[R^E_{x_0} + \frac{1}{2}\mathrm{Tr}(R^{T^{(1,0)}X}_{x_0})\big] - \frac{1}{4}\,{}^c\big(dT_{as}\big)_{x_0} - \frac{1}{8}\big|T_{as}\big|^2_{x_0}\Big\} + O(t^3). \tag{3.5.9}$$

Clearly by (3.5.1) we have

$$\nabla_{t,e_i} = \nabla_{e_i} + t\Gamma^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_{tZ}(e_i) - \frac{t}{2}k^{-1}(tZ)(\nabla_{e_i}k)(tZ), \tag{3.5.10}$$

where $\Gamma^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}$ is induced by $\Gamma^{B,\Lambda^{0,\bullet}}$, Γ^L and Γ^E. Recall that by [30, (1.2.30)]:

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E})_{x_0}(e_i)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E})_{x_0}(\mathcal{R},e_i)\frac{Z^\alpha}{\alpha!}. \tag{3.5.11}$$

Therefore,

$$\begin{aligned} \Gamma^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_Z(e_i) =& \frac{1}{2}R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_{x_0}(\mathcal{R},e_i) + \frac{1}{3}\sum_{|\alpha|=1}(\partial_j R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E})_{x_0}(\mathcal{R},e_i)Z_j \\ &+ \frac{1}{4}\sum_{|\alpha|=2}(\partial^\alpha R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E})_{x_0}(\mathcal{R},e_i)\frac{Z^\alpha}{\alpha!} + O(\big|Z\big|^4), \end{aligned} \tag{3.5.12}$$

which implies

$$\begin{aligned} t\Gamma^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_{tZ}(e_i) =& \frac{t^2}{2}R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_{x_0}(\mathcal{R},e_i) + \frac{t^3}{3}\sum_{|\alpha|=1}(\partial_j R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E})_{x_0}(\mathcal{R},e_i)Z_j \\ &+ \frac{t^4}{4}\sum_{|\alpha|=2}(\partial^\alpha R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E})_{x_0}(\mathcal{R},e_i)\frac{Z^\alpha}{\alpha!} + O(t^5). \end{aligned} \tag{3.5.13}$$

Substituting (3.2.19) and (3.5.13) into (3.5.10), we have

$$\begin{aligned}\nabla_{t,e_i} =&\nabla_{e_i} + \frac{t^2}{2} R^{B,\Lambda^{0,\bullet}\otimes L^p\otimes E}_{x_0}(\mathcal{R}, e_i) - \frac{t}{2} k^{-1}(tZ)(\nabla_{e_i}k)(tZ) \\ &+ \frac{t}{3}\sum_{|\alpha|=1}(\partial_j R^L)_{x_0}(\mathcal{R}, e_i)Z_j + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(\mathcal{R}, e_i)\frac{Z^\alpha}{\alpha!} + O(t^3).\end{aligned} \tag{3.5.14}$$

That is

$$\begin{aligned}\nabla_{t,e_i} =&\nabla_{e_i} + \frac{t^2}{2} R^E_{x_0}(\mathcal{R}, e_i) + \frac{t^2}{2} R^{B,\Lambda^{0,\bullet}}_{x_0}(\mathcal{R}, e_i) + \frac{1}{2} R^L_{x_0}(\mathcal{R}, e_i) \\ &+ \frac{t}{3}\sum_{|\alpha|=1}(\partial_j R^L)_{x_0}(\mathcal{R}, e_i)Z_j + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(\mathcal{R}, e_i)\frac{Z^\alpha}{\alpha!} \\ &- \frac{t}{2} k^{-1}(tZ)(\nabla_{e_i}k)(tZ) + O(t^3).\end{aligned} \tag{3.5.15}$$

From [29, (2.8)] we deduce

$$-\frac{t}{2} k^{-1}(tZ)(\nabla_{e_i}k)(tZ) = -\frac{t^2}{6}\Big\langle R^{TX}_{x_0}(e_j, e_i)e_j, \mathcal{R}\Big\rangle + O(t^3). \tag{3.5.16}$$

Substituting (3.5.16) into (3.5.15), we obtain

$$\begin{aligned}\nabla_{t,e_i} =&\nabla_{e_i} + \Big(\frac{1}{2} R^L_{x_0} + \frac{t}{3}(\partial_k R^L)_{x_0} Z_k + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2} R^E_{x_0} + \frac{t^2}{2} R^{B,\Lambda^{0,\bullet}}_{x_0}\Big)(\mathcal{R}, e_i) \\ &- \frac{t^2}{6}\Big\langle R^{TX}_{x_0}(e_i, e_j)\mathcal{R}, e_j\Big\rangle + O(t^3) \qquad (3.5.17) \\ =&\nabla_{0,e_i} + \frac{t}{3}(\partial_k R^L)_{x_0}(\mathcal{R}, e_i)Z_k + O(t^3) \\ &+ \frac{t^2}{2}\Big[\Big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\Big)(\mathcal{R}, e_i) - \frac{1}{3}\Big\langle R^{TX}_{x_0}(e_i, e_j)\mathcal{R}, e_j\Big\rangle\Big].\end{aligned}$$

Substituting (3.5.17) and [30, (4.1.102)] into (3.5.7), we get

$$\begin{aligned}
\mathcal{L}^t = &- \Big[\delta_{ij} - \frac{t^2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle + O(t^3)\Big] \\
&\times \Big\{\nabla_{0,e_i} + \frac{t}{3}(\partial_p R^L)_{x_0}(\mathcal{R}, e_i)Z_p + O(t^3) \\
&\quad + \frac{t^2}{2}\Big[\big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_i) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_i, e_p)\mathcal{R}, e_p\big\rangle\Big]\Big\} \\
&\times \Big\{\nabla_{0,e_j} + \frac{t}{3}(\partial_l R^L)_{x_0}(\mathcal{R}, e_j)Z_l + O(t^3) \\
&\quad + \frac{t^2}{2}\Big[\big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_j) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_j, e_l)\mathcal{R}, e_l\big\rangle\Big]\Big\} \\
&+ t\Big[\delta_{ij} - \frac{t^2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle + O(t^3)\Big] \\
&\times \Big[\frac{t}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)e_j + R^{TX}_{x_0}(\mathcal{R}, e_j)e_i, e_k\big\rangle + O(t^3)\Big] \\
&\times \Big\{\nabla_{0,e_k} + \frac{t}{3}(\partial_l R^L)_{x_0}(\mathcal{R}, e_k)Z_l + O(t^3) \\
&\quad + \frac{t^2}{2}\Big[\big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_k) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_k, e_l)\mathcal{R}, e_l\big\rangle\Big]\Big\} + O(t^3) \\
= &- (\nabla_{0,e_j})^2 - \frac{t}{3}\Big[(\partial_l R^L)_{x_0}(\mathcal{R}, e_j)Z_l\nabla_{0,e_j} + \nabla_{0,e_j}\cdot(\partial_l R^L)_{x_0}(\mathcal{R}, e_j)Z_l\Big] \\
&- t^2\Big\{\frac{1}{2}\Big[\big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_j) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_j, e_l)\mathcal{R}, e_l\big\rangle\Big]\nabla_{0,e_j} \\
&\quad + \frac{1}{2}\nabla_{0,e_j}\Big[\big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_j) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_j, e_l)\mathcal{R}, e_l\big\rangle\Big] \\
&\quad + \frac{1}{9}\big[(\partial_k R^L)_{x_0}(\mathcal{R}, e_j)Z_k\big] \times \big[(\partial_l R^L)_{x_0}(\mathcal{R}, e_j)Z_l\big] \\
&\quad - \frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle\nabla_{0,e_i}\nabla_{0,e_j} - \frac{2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_j, e_k\big\rangle\nabla_{0,e_k}\Big\} + O(t^3) \\
= &- (\nabla_{0,e_j})^2 - \frac{t}{3}\Big[2(\partial_l R^L)_{x_0}(\mathcal{R}, e_j)Z_l\nabla_{0,e_j} + (\partial_j R^L)_{x_0}(\mathcal{R}, e_j)\Big] \\
&- t^2\Big\{\Big[\big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_j) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_j, e_l)\mathcal{R}, e_l\big\rangle\Big]\nabla_{0,e_j} \\
&\quad + \frac{1}{2}\Big[\nabla_{0,e_j}, \big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_j) - \frac{1}{3}\big\langle R^{TX}_{x_0}(e_j, e_l)\mathcal{R}, e_l\big\rangle\Big] \\
&\quad + \frac{1}{9}\sum_{i=1}^{2n}\Big[\sum_{j=1}^{2n}(\partial_j R^L)_{x_0}(\mathcal{R}, e_i)Z_j\Big]^2 \\
&- \frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle\nabla_{0,e_i}\nabla_{0,e_j} - \frac{2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_j, e_k\big\rangle\nabla_{0,e_k}\Big\} + O(t^3).
\end{aligned}$$

That is

$$\begin{aligned}\mathcal{L}^t =& -(\nabla_{0,e_j})^2 - \frac{t}{3}\Big[2(\partial_l R^L)_{x_0}(\mathcal{R}, e_j) Z_l \nabla_{0,e_j} + (\partial_j R^L)_{x_0}(\mathcal{R}, e_j)\Big] \\ &+ t^2 \Big\{ \frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle \nabla_{0,e_i}\nabla_{0,e_j} - \frac{1}{4}\Big[\nabla_{0,e_j}, \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}\Big] \\ &\quad + \frac{1}{6}\big\langle R^{TX}_{x_0}(e_j, e_l)e_j, e_l\big\rangle - \frac{1}{9}\sum_{i=1}^{2n}\Big[\sum_{j=1}^{2n}(\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j\Big]^2 \\ &\quad + \Big[\frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_l)e_l, e_j\big\rangle - \big(R^{B,\Lambda^{0,\bullet}}_{x_0} + R^E_{x_0} + \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}\big)(\mathcal{R}, e_j)\Big]\nabla_{0,e_j}\Big\}. \\ &\quad + O(t^3).\end{aligned} \tag{3.5.18}$$

Clearly,

$$\begin{aligned}&\frac{1}{12}\Big[(\nabla_{0,e_j})^2, \big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\big\rangle\Big] \\ =&\frac{1}{6}\Big[\nabla_{0,e_j}, \big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\big\rangle\Big]\nabla_{0,e_j} + \frac{1}{12}\Big[\nabla_{0,e_j}, \big[\nabla_{0,e_j}, \langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\rangle\big]\Big] \\ =& -\frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_k)e_k, e_j\big\rangle\nabla_{0,e_j} + \frac{1}{6}\big\langle R^{TX}_{x_0}(e_j, e_i)e_j, e_i\big\rangle.\end{aligned} \tag{3.5.19}$$

That is

$$\begin{aligned}&\frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_k)e_k, e_j\big\rangle\nabla_{0,e_j} + \frac{1}{6}\big\langle R^{TX}_{x_0}(e_j, e_i)e_j, e_i\big\rangle \\ =&\frac{2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_k)e_k, e_j\big\rangle\nabla_{0,e_j} + \frac{1}{12}\Big[(\nabla_{0,e_j})^2, \big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\big\rangle\Big].\end{aligned} \tag{3.5.20}$$

Substituting (3.5.8), (3.5.9) and (3.5.18) into (3.5.6) yields (3.5.5) with $\mathcal{O}_1, \mathcal{O}_2$ given by (3.5.4). The proof of Theorem 3.34 is complete. □

3.5.2 The new formula for $\mathbf{b}_1$

By Proposition 3.7, (3.3.1), (3.5.4) and (3.2.29),

$$\mathcal{O}_1 = \mathcal{O}_1' - 8\sqrt{-1}\pi\Big\langle(\nabla^B_{\mathcal{R}}\mathbf{J})\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_k}\Big\rangle d\overline{z}_k \wedge i_{\frac{\partial}{\partial \overline{z}_j}}. \tag{3.5.21}$$

We have the following analogue of [29, Theorem 2.3].

Theorem 3.35. *The following relation holds:*

$$P^N \mathcal{O}_1 P^N = 0. \tag{3.5.22}$$

Proof. Set

$$\mathcal{J} = -2\pi\sqrt{-1}\mathbf{J}. \tag{3.5.23}$$

By (3.5.2) and (3.2.29),

$$\begin{aligned}\mathcal{O}_1'(Z) =& -\frac{2}{3}\Big\langle(\nabla^B_Z\mathcal{J})\mathcal{R}, f_j\Big\rangle\nabla_{0,f_j} - \frac{1}{3}\Big\langle(\nabla^B_{f_j}\mathcal{J})\mathcal{R}, f_j\Big\rangle \\ =& -\frac{4}{3}\Big\langle(\nabla^B_Z\mathcal{J})\mathcal{R}, \frac{\partial}{\partial\xi_j}\Big\rangle\nabla_{0,\frac{\partial}{\partial\bar{\xi}_j}} - \frac{4}{3}\Big\langle(\nabla^B_Z\mathcal{J})\mathcal{R}, \frac{\partial}{\partial\bar{\xi}_j}\Big\rangle\nabla_{0,\frac{\partial}{\partial\xi_j}}.\end{aligned} \tag{3.5.24}$$

From (3.2.29) and (3.4.203), we find that

$$\mathcal{O}_1'(Z) = -\frac{2}{3}\Big[\Big\langle(\nabla^B_Z\mathcal{J})\mathcal{R}, \frac{\partial}{\partial\xi_j}\Big\rangle b_j^+ - b_j\Big\langle(\nabla^B_Z\mathcal{J})\mathcal{R}, \frac{\partial}{\partial\bar{\xi}_j}\Big\rangle\Big]. \tag{3.5.25}$$

By Theorem 3.31 and (3.4.210), any polynomial $g(\xi,\overline{\xi})$ in $\xi,\overline{\xi}$ satisfies

$$\mathcal{P}b^\alpha g(\xi,\overline{\xi})\mathcal{P} = 0, \quad \text{for } |\alpha| > 0. \tag{3.5.26}$$

By (3.5.25) and (3.5.26), we get

$$\mathcal{P}\mathcal{O}_1'\mathcal{P} = 0. \tag{3.5.27}$$

Now (3.5.22) follows from (3.2.29), (3.4.212), (3.5.21) and (3.5.27). □

Substituting (3.5.22) into (3.4.224) we find

$$\begin{aligned}\mathcal{F}_2 =&(\mathcal{L}^0_2)^{-1}P^{N^\perp}\mathcal{O}_1(\mathcal{L}^0_2)^{-1}P^{N^\perp}\mathcal{O}_1P^N - (\mathcal{L}^0_2)^{-1}P^{N^\perp}\mathcal{O}_2P^N \\ &+ P^N\mathcal{O}_1(\mathcal{L}^0_2)^{-1}P^{N^\perp}\mathcal{O}_1(\mathcal{L}^0_2)^{-1}P^{N^\perp} - P^N\mathcal{O}_2(\mathcal{L}^0_2)^{-1}P^{N^\perp} \\ &+ (\mathcal{L}^0_2)^{-1}P^{N^\perp}O_1P^N\mathcal{O}_1(\mathcal{L}^0_2)^{-1}P^{N^\perp} - P^N\mathcal{O}_1P^{N^\perp}(\mathcal{L}^0_2)^{-2}\mathcal{O}_1P^N,\end{aligned} \tag{3.5.28}$$

and

$$\mathbf{b}_1(x_0) = F_2(0,0). \tag{3.5.29}$$

We only need to compute the first two terms and the last two terms in (3.5.28), since the third and fourth term in (3.5.28) are adjoint of the first two terms by Theorem 3.34.

3.6 Calculation of the coefficient b_1

In this Section, we calculate term by term of the formula (3.5.28) of the second coefficient $\mathbf{b}_1$. Subsection 3.6.1 is devoted to a formula for the scalar curvature r^X. In subsection 3.6.2, we calculate the terms in (3.5.28) containing the factor $\mathcal{O}_1$. In subsection 3.6.3, we calculate the rest terms in (3.5.28) and then obtain the formula (3.1.19).

3.6.1 A formula for the scalar curvature r^X

Before our calculation, we establish the relation between the scalar curvature r^X and $|\nabla^X \mathbf{J}|^2$, which is exactly the same as [31, Lemma 2.2]. Here we use the coordinate $\xi_1, \dots, \xi_n$.

Lemma 3.36.

$$r^X = 8\Big\langle R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j})\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_i}\Big\rangle - \frac{1}{4}\big|\nabla^X \mathbf{J}\big|^2. \tag{3.6.1}$$

Proof. We give the proof for the sake of completeness. By the definition of r^X,

$$\begin{aligned} r^X =& -\Big\langle R^{TX}(f_i, f_j)f_i, f_j\Big\rangle = -4\Big\langle R^{TX}(\frac{\partial}{\partial \xi_i}, f_j)\frac{\partial}{\partial \overline{\xi}_i}, f_j\Big\rangle \\ =& 8\Big\langle R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j})\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_i}\Big\rangle - 8\Big\langle R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle. \end{aligned} \tag{3.6.2}$$

By (3.1.13),

$$\Big\langle R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle = \frac{\sqrt{-1}}{2}\Big\langle \big[R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}), \mathbf{J}\big]\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle. \tag{3.6.3}$$

From (3.2.25) and anti-symmetry of $(\nabla^X\nabla^X \mathbf{J})_{(\bullet,\bullet)}$, we get

$$\Big\langle \big[R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}), \mathbf{J}\big]\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle = 2\Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})}\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle. \tag{3.6.4}$$

By (3.2.26),

$$\Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})}\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle = \Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})}\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle - \Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j})}\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_i}\Big\rangle. \tag{3.6.5}$$

By (3.2.25) and (3.2.29), we find that

$$2\sqrt{-1}\Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})}\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle = \Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}} \mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_i}} \mathbf{J})\frac{\partial}{\partial \overline{\xi}_j}\Big\rangle. \tag{3.6.6}$$

Similarly,

$$2\sqrt{-1}\Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j})}\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_i}\Big\rangle = \Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}} \mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}} \mathbf{J})\frac{\partial}{\partial \overline{\xi}_i}\Big\rangle. \tag{3.6.7}$$

Substituting (3.6.6), (3.6.7) into (3.6.5), one immediately gets

$$\begin{aligned} &2\sqrt{-1}\Big\langle \big(\nabla^X\nabla^X \mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})}\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle \\ =&\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}} \mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_i}} \mathbf{J})\frac{\partial}{\partial \overline{\xi}_j}\Big\rangle - \Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}} \mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}} \mathbf{J})\frac{\partial}{\partial \overline{\xi}_i}\Big\rangle. \end{aligned} \tag{3.6.8}$$

By the definition of $|\nabla^X \mathbf{J}|^2$ and (3.2.29),

$$|\nabla^X \mathbf{J}|^2 = \Big\langle (\nabla^X_{f_i}\mathbf{J})f_j, (\nabla^X_{f_i}\mathbf{J})f_j \Big\rangle = 8\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}}\mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_j} \Big\rangle. \tag{3.6.9}$$

Thus,

$$\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}}\mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_j} \Big\rangle = \frac{1}{8}|\nabla^X \mathbf{J}|^2. \tag{3.6.10}$$

By (3.2.24) and (3.2.29),

$$\begin{aligned}
&\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}}\mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_i} \Big\rangle = 2\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}}\mathbf{J})\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k} \Big\rangle \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle \\
=&2\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_j}}\mathbf{J})\frac{\partial}{\partial \xi_k} - (\nabla^X_{\frac{\partial}{\partial \xi_k}}\mathbf{J})\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_i} \Big\rangle \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_k}, \frac{\partial}{\partial \overline{\xi}_i} \Big\rangle \\
=&\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_j}}\mathbf{J})\frac{\partial}{\partial \xi_k}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_k}, \Big\rangle - \Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_k}}\mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_k} \Big\rangle.
\end{aligned} \tag{3.6.11}$$

Combining (3.6.10) and (3.6.11), we find that

$$\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}}\mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_i} \Big\rangle = \frac{1}{2}\Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_i}}\mathbf{J})\frac{\partial}{\partial \xi_j}, (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial \overline{\xi}_j} \Big\rangle = \frac{1}{16}|\nabla^X \mathbf{J}|^2. \tag{3.6.12}$$

By (3.6.3), (3.6.4), (3.6.8), (3.6.10) and (3.6.12), we obtain

$$\Big\langle R^{TX}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j} \Big\rangle = \frac{1}{32}\big|\nabla^X \mathbf{J}\big|^2. \tag{3.6.13}$$

Now (3.6.1) follows from (3.6.2) and (3.6.13). □

We are now ready to compute the terms in the expression (3.5.28).

Lemma 3.37. *For every 2-form A, we have*

$$\begin{aligned}
{}^c(A)\cdot I_{\det(\overline{W}^*)\otimes E} =&\Big[-2A(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}) + 4\sum_{j=1}^{q}\sum_{k=q+1}^{n} A(\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k})d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \\
&+ 4\sum_{j,k=1}^{q} A(\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}) i_{\frac{\partial}{\partial \overline{\xi}_j}} i_{\frac{\partial}{\partial \overline{\xi}_k}} + \sum_{j,k=q+1}^{n} A(\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k})d\overline{\xi}_j \wedge d\overline{\xi}_k\Big] I_{\det(\overline{W}^*)\otimes E}.
\end{aligned} \tag{3.6.14}$$

If A is compatible with the complex structure J, then

$${}^c(A)\cdot I_{\det(\overline{W}^*)\otimes E} = \Big[-2A(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}) + 4\sum_{j=1}^{q}\sum_{k=q+1}^{n} A(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_k})d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}}\Big] I_{\det(\overline{W}^*)\otimes E}. \tag{3.6.15}$$

Proof. One easily get the result (3.6.14) from (3.3.1). □

3.6.2 The term in b_1 containing the factor $\mathscr{O}_1$

By (3.2.29), (3.4.210), (3.4.212), (3.5.21) and (3.5.25), we know that

$$\begin{aligned}
&\mathscr{O}_1 P^N(Z,Z') \\
=&\Big[-\frac{2\sqrt{-1}}{3} b_i b_j \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}} \mathbf{J})\overline{\xi}', \frac{\partial}{\partial \overline{\xi}_i} \Big\rangle - \frac{4\pi\sqrt{-1}}{3} b_i \Big\langle (\nabla^X_{\overline{\xi}'} \mathbf{J})\overline{\xi}', \frac{\partial}{\partial \overline{\xi}_i} \Big\rangle \\
&- 4\sqrt{-1} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\frac{\partial}{\partial \overline{\xi}_m}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} (b_m + 2\pi \overline{\xi}'_m) \\
&- 8\pi\sqrt{-1} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\xi} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \Big] \mathcal{P}(Z,Z') I_{\det(\overline{W}^*)\otimes E}.
\end{aligned} \tag{3.6.16}$$

From Theorem 3.31 and (3.6.16),

$$\begin{aligned}
&\big((\mathscr{L}^0_2)^{-1} P^{N^\perp} \mathscr{O}_1 P^N\big)(Z,Z') \\
=& -\sqrt{-1} \Big[\frac{b_i b_j}{12\pi} \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{\xi}_j}} \mathbf{J})\overline{\xi}', \frac{\partial}{\partial \overline{\xi}_i} \Big\rangle + \frac{b_i}{3} \Big\langle (\nabla^X_{\overline{\xi}'} \mathbf{J})\overline{\xi}', \frac{\partial}{\partial \overline{\xi}_i} \Big\rangle \\
&+ \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\frac{\partial}{\partial \overline{\xi}_m}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} (\frac{b_m}{3\pi} + \overline{\xi}'_m) \\
&+ \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\xi} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \Big] \mathcal{P}(Z,Z') I_{\det(\overline{W}^*)\otimes E}.
\end{aligned} \tag{3.6.17}$$

Therefore,

$$\begin{aligned}
&\Big((\mathscr{L}^0_2)^{-1} P^{N^\perp} \mathscr{O}_1 P^N\Big)(0,Z') \\
=& -\frac{\sqrt{-1}}{3} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\overline{\xi}'} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \mathcal{P}(0,Z') I_{\det(\overline{W}^*)\otimes E},
\end{aligned} \tag{3.6.18}$$

and

$$\begin{aligned}
&\Big((\mathscr{L}^0_2)^{-1} P^{N^\perp} \mathscr{O}_1 P^N\Big)(Z,0) \\
=& -\frac{2\sqrt{-1}}{3} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\overline{\xi}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \mathcal{P}(Z,0) I_{\det(\overline{W}^*)\otimes E} \\
&- \sqrt{-1} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\xi} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \mathcal{P}(Z,0) I_{\det(\overline{W}^*)\otimes E}.
\end{aligned} \tag{3.6.19}$$

By taking adjoint of (3.6.18) and (3.6.19), we find that

$$\begin{aligned}
&\Big(P^N \mathscr{O}_1 (\mathscr{L}^0_2)^{-1} P^{N^\perp}\Big)(Z',0) \\
=& \frac{\sqrt{-1}}{3} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\xi'} \mathbf{J}) \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k} \Big\rangle \mathcal{P}(Z',0) I_{\det(\overline{W}^*)\otimes E} d\xi_j \wedge i_{\frac{\partial}{\partial \overline{\xi}_k}},
\end{aligned} \tag{3.6.20}$$

and

$$\begin{aligned}&\Big(P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp}\Big)(0,Z)\\ =&\frac{2\sqrt{-1}}{3}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big\langle(\nabla^B_\xi\mathbf{J})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_k}\Big\rangle\mathcal{P}(0,Z)I_{\det(\overline{W}^*)\otimes E}d\xi_j\wedge i_{\frac{\partial}{\partial\overline{\xi}_k}}\\ &+\sqrt{-1}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big\langle(\nabla^B_{\overline{\xi}}\mathbf{J})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_k}\Big\rangle\mathcal{P}(0,Z)I_{\det(\overline{W}^*)\otimes E}d\xi_j\wedge i_{\frac{\partial}{\partial\overline{\xi}_k}}.\end{aligned}\tag{3.6.21}$$

By (3.2.29), (3.6.19), (3.6.21) and $\int_{\mathbb{C}}|\xi|^2e^{-\pi|\xi|^2}=\frac{1}{\pi}$, we obtain

$$\begin{aligned}&\Big(P^N\mathcal{O}_1P^{N^\perp}(\mathcal{L}_2^0)^{-2}\mathcal{O}_1P^N\Big)(0,0)\\ =&\frac{4}{9\pi}\sum_{m=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big|\Big\langle(\nabla^B_{\frac{\partial}{\partial\xi_m}}\mathbf{J})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_k}\Big\rangle\Big|^2I_{\det(\overline{W}^*)\otimes E}\\ &+\frac{1}{\pi}\sum_{m=1}^{n}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big|\Big\langle(\nabla^B_{\frac{\partial}{\partial\overline{\xi}_m}}\mathbf{J})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_k}\Big\rangle\Big|^2I_{\det(\overline{W}^*)\otimes E}\\ =&\frac{1}{72\pi}\Big(\big|\nabla^B\mathbf{J}\big|^2+10\sum_{i,j,k=1}^{n}\big|\big\langle S^B(\overline{u}_i)u_j,u_k\big\rangle\big|^2\Big)I_{\det(\overline{W}^*)\otimes E}.\end{aligned}\tag{3.6.22}$$

By (3.6.18) and (3.6.20),

$$\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N\mathcal{O}_1(\mathcal{L}_2^0)^{-1}P^{N^\perp}\Big)(0,0).\tag{3.6.23}$$

$$=\frac{1}{9\pi}\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}\Big\langle(\nabla^B_{\frac{\partial}{\partial\overline{\xi}_m}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_l}\Big\rangle\Big\langle(\nabla^B_{\frac{\partial}{\partial\overline{\xi}_m}}\mathbf{J})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_k}\Big\rangle d\overline{\xi}_l\wedge i_{\frac{\partial}{\partial\xi_i}}I_{\det(\overline{W}^*)\otimes E}d\xi_j\wedge i_{\frac{\partial}{\partial\overline{\xi}_k}}.$$

Let $h(Z)$ (resp. $F(Z)$) be homogenous polynomials in Z with degree 1 (resp. 2), then by (3.4.210) and Theorem 3.31,

$$\begin{aligned}\big(\mathcal{L}_0^{-1}\mathcal{P}^\perp hb_j\mathcal{P}\big)(0,0)=&\big(\mathcal{L}_0^{-1}\mathcal{P}^\perp b_jh\mathcal{P}\big)(0,0)=-\frac{1}{2\pi}\frac{\partial h}{\partial\xi_j},\\ \big(\mathcal{L}_0^{-1}\mathcal{P}^\perp F\mathcal{P}\big)(0,0)=&-\frac{1}{4\pi^2}\frac{\partial^2F}{\partial\xi_j\partial\overline{\xi}_j}.\end{aligned}\tag{3.6.24}$$

From (3.4.210) and Theorem 3.31, one verifies directly the following relations. For $1\leqslant j\leqslant q, q+1\leqslant k\leqslant n$,

$$\begin{aligned}\Big((\mathcal{L}_2^0)^{-1}hb_md\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\xi_j}}P^N\Big)(0,0)=&\frac{1}{12\pi}\frac{\partial h}{\partial\xi_m}d\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\xi_j}}I_{\det(\overline{W}^*)\otimes E},\\ \Big((\mathcal{L}_2^0)^{-1}Fd\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\xi_j}}P^N\Big)(0,0)=&\frac{1}{24\pi^2}\frac{\partial^2F}{\partial\xi_i\partial\overline{\xi}_i}d\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\xi_j}}I_{\det(\overline{W}^*)\otimes E}.\end{aligned}\tag{3.6.25}$$

Moreover, we have for $1 \leqslant i,j \leqslant q$ and $q+1 \leqslant k,l \leqslant n$,

$$\Big((\mathcal{L}_2^0)^{-1} F d\overline{\xi}_k \wedge d\overline{\xi}_l i_{\frac{\partial}{\partial \xi_i}} i_{\frac{\partial}{\partial \xi_j}} P^N\Big)(0,0) = \frac{1}{80\pi^2} \frac{\partial^2 F}{\partial \xi_m \partial \overline{\xi}_m} d\overline{\xi}_k \wedge d\overline{\xi}_l i_{\frac{\partial}{\partial \xi_i}} i_{\frac{\partial}{\partial \xi_j}} I_{\det(\overline{W}^*)\otimes E}. \tag{3.6.26}$$

By (3.2.29), (3.5.21) and (3.6.19), we get

$$\begin{aligned}
&\Big((\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} \mathcal{O}_1 P^N\Big)(0,0) \\
&= -\frac{16}{3}\pi \Big\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \Big[\Big\langle (\nabla^B_{\mathcal{R}} \mathbf{J}) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_l} \Big\rangle d\overline{z}_l \wedge i_{\frac{\partial}{\partial \overline{z}_i}} \\
&\quad \times \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\xi} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \Big] \mathcal{P} \Big\}(0,0) I_{\det(\overline{W}^*)\otimes E} \\
&\quad - 8\pi \Big\{ (\mathcal{L}_2^0)^{-1} \mathcal{P}^{N^\perp} \Big[\Big\langle (\nabla^B_{\mathcal{R}} \mathbf{J}) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_l} \Big\rangle d\overline{z}_l \wedge i_{\frac{\partial}{\partial \overline{z}_i}} \\
&\quad \times \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big\langle (\nabla^B_{\xi} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}} \Big] \mathcal{P} \Big\}(0,0) I_{\det(\overline{W}^*)\otimes E} \qquad (3.6.27) \\
&= -\frac{4}{3\pi} \sum_{m=1}^{n} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big| \Big\langle (\nabla^X_{\frac{\partial}{\partial \xi_m}} \mathbf{J}) \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k} \Big\rangle \Big|^2 I_{\det(\overline{W}^*)\otimes E} + I_1 \\
&\quad - \frac{2}{\pi} \sum_{m=1}^{n} \sum_{j=1}^{q} \sum_{k=q+1}^{n} \Big| \Big\langle (\nabla^B_{\frac{\partial}{\partial \overline{\xi}_m}} \mathbf{J}) \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k} \Big\rangle \Big|^2 I_{\det(\overline{W}^*)\otimes E} + I_2 \\
&= -\frac{1}{24\pi} \Big(\big| \nabla^B \mathbf{J} \big|^2 + 4 \sum_{i,j,k=1}^{n} \big| \big\langle S^B(\overline{u}_i) u_j, u_k \big\rangle \big|^2 \Big) I_{\det(\overline{W}^*)\otimes E} + I_1 + I_2,
\end{aligned}$$

with

$$\begin{aligned}
I_1 = -\frac{1}{15\pi} \sum_{i,j=1}^{q} \sum_{k,l=q+1}^{n} &\Big\langle (\nabla^B_{\frac{\partial}{\partial \xi_m}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_l} \Big\rangle \Big\langle (\nabla^B_{\frac{\partial}{\partial \xi_m}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle \\
&\times d\overline{\xi}_k \wedge d\overline{\xi}_l \wedge i_{\frac{\partial}{\partial \xi_i}} i_{\frac{\partial}{\partial \xi_j}} I_{\det(\overline{W}^*)\otimes E}
\end{aligned} \tag{3.6.28}$$

and

$$\begin{aligned}
I_2 = -\frac{1}{10\pi} \sum_{i,j=1}^{q} \sum_{k,l=q+1}^{n} &\Big\langle (\nabla^B_{\frac{\partial}{\partial \overline{\xi}_m}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_l} \Big\rangle \Big\langle (\nabla^B_{\frac{\partial}{\partial \xi_m}} \mathbf{J}) \frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k} \Big\rangle \\
&\times d\overline{\xi}_k \wedge d\overline{\xi}_l \wedge i_{\frac{\partial}{\partial \xi_i}} i_{\frac{\partial}{\partial \xi_j}} I_{\det(\overline{W}^*)\otimes E}.
\end{aligned} \tag{3.6.29}$$

3.6.3 The term in b_1 containing the factor $\mathcal{O}_2$

Before computing $\big((\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P^N\big)(0,0)$, we first calculate $\big(\mathcal{L}_0^{-1} \mathcal{P}^\perp \mathcal{O}_2' \mathcal{P}\big)(0,0)$. The following result is due to Ma and Marinescu, see [31, Lemma 2.1].

Lemma 3.38. *The following relation holds for the operator $\mathcal{O}_2'$:*

$$\begin{aligned}\mathcal{O}_2'\mathcal{P} =&\Big\{\frac{1}{3}b_ib_j\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle+\frac{1}{2}b_i\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i}\Big)\frac{Z^\alpha}{\alpha!}\\&+\frac{4}{3}b_j\Big[\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle-\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\bar{\xi}_i},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle\Big]+R^E\Big(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i}\Big)b_i\\&+\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle+4\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j})\frac{\partial}{\partial\bar{\xi}_i},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle\Big\}\mathcal{P}\\&+\Big(-\frac{1}{3}\mathcal{L}_0\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_j})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle+\frac{1}{9}\big|(\nabla_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2\Big)\mathcal{P}.\end{aligned}\tag{3.6.30}$$

Proof. We give the proof for the sake of completeness. From (3.4.203) we have

$$\begin{aligned}&\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_j\rangle\nabla_{0,e_i}\nabla_{0,e_j}\\=&\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\xi_j}\rangle b_i^+b_j^+-\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\rangle b_i^+b_j\\&-\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\xi_j}\rangle b_ib_j^++\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\rangle b_ib_j\\=&\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\rangle b_ib_j+\frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\xi_j}\rangle b_i^+b_j^+\\&-\frac{2}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\xi_j}\rangle b_ib_j^+-\frac{4\pi}{3}\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\xi_i}\rangle.\end{aligned}\tag{3.6.31}$$

Since

$$\sum_{j=1}^{2n}\big(\partial_jR^L\big)_{x_0}(\mathcal{R},e_i)Z_j=\Big\langle(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R},e_i\Big\rangle,\tag{3.6.32}$$

then

$$\sum_{i=1}^{2n}\Big[\sum_{j=1}^{2n}\big(\partial_jR^L\big)_{x_0}(\mathcal{R},e_i)Z_j\Big]^2=-\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2.\tag{3.6.33}$$

Set

$$\begin{aligned}\mathcal{O}_{2,1}' =&\frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})b_i-\frac{1}{2}\frac{\partial}{\partial\bar{\xi}_i}\Big(\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\xi_i})\Big)\\&-\frac{1}{2}\frac{\partial}{\partial\xi_i}\Big[\big(\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}+R^E_{x_0}\big)(\mathcal{R},\frac{\partial}{\partial\xi_i})\Big];\\\mathcal{O}_{2,2}' =&\frac{1}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle b_ib_j-\frac{2}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R},e_j)e_j,\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle b_i.\end{aligned}$$

Then

$$\begin{aligned}\mathscr{O}'_2 =&\mathscr{O}'_{2,1} + \mathscr{O}'_{2,2} - \frac{1}{3}\Big[\mathscr{L}_0, \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i}\big\rangle\Big] + R^E_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})b_i\\
&+ \Big[\frac{2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_j, \frac{\partial}{\partial \overline{\xi}_i}\big\rangle - \big(\frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!} + R^E_{x_0}\big)(\mathcal{R}, \frac{\partial}{\partial \xi_i})\Big]b_i^+\\
&+ \frac{1}{3}\Big[\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \xi_i})\mathcal{R}, \frac{\partial}{\partial \xi_j}\big\rangle b_i^+ b_j^+ - 2\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \xi_j}\big\rangle b_i b_j^+\\
&\quad - 4\pi\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \xi_i}\big\rangle\Big] + \frac{1}{9}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2. \qquad (3.6.34)\end{aligned}$$

Clearly,

$$\begin{aligned}\mathscr{O}'_{2,2} =&\frac{1}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle b_i b_j\\
&- \frac{4}{3}\Big[\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle + \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \xi_i})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big]b_j\\
=&\frac{b_i}{3}\Big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle b_j - \frac{1}{3}\Big[b_i, \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big]b_j\\
&- \frac{4}{3}b_j\Big[\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle + \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \xi_i})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big]\\
&+ \frac{4}{3}\Big[b_j, \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle + \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \xi_i})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big].\\
=&\frac{1}{3}b_i b_j\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle \qquad (3.6.35)\\
&+ \frac{4}{3}b_j\Big[\big\langle R^{TX}_{x_0}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle - \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \xi_i})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big]\\
&+ \frac{4}{3}\Big[\big\langle R^{TX}_{x_0}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle - \big\langle R^{TX}_{x_0}(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_i})\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big]\\
&- \frac{8}{3}\big\langle R^{TX}_{x_0}(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_i})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\\
=&\frac{1}{3}b_i b_j\big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle + 4\big\langle R^{TX}_{x_0}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\\
&+ \frac{4}{3}b_j\Big[\big\langle R^{TX}_{x_0}(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle - \big\langle R^{TX}_{x_0}(\mathcal{R}, \frac{\partial}{\partial \xi_i})\frac{\partial}{\partial \overline{\xi}_i}, \frac{\partial}{\partial \overline{\xi}_j}\big\rangle\Big].\end{aligned}$$

Recall that by [31, (2.7)],

$$\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}(\mathcal{R}, \cdot)\frac{Z^\alpha}{\alpha!} = \frac{1}{2}\big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \cdot\big\rangle + \frac{1}{6}\big\langle R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\mathcal{R}, \cdot\big\rangle. \qquad (3.6.36)$$

Then

$$
\begin{aligned}
\mathcal{O}'_{2,1} =& \frac{1}{2}b_i\Big[\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i})\Big] - \frac{1}{2}\Big[b_i,\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i})\Big] \\
& - \frac{1}{4}\frac{\partial}{\partial\overline{\xi}_i}\Big(\big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\frac{\partial}{\partial\xi_i}\big\rangle + \frac{1}{3}\big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\frac{\partial}{\partial\xi_i}\big\rangle\Big) \\
& - \frac{1}{4}\frac{\partial}{\partial\xi_i}\Big(\big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\big\rangle + \frac{1}{3}\big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\big\rangle\Big) \\
=& \frac{1}{2}b_i\Big[\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i})\Big] \\
& + \frac{1}{4}\Big(\frac{\partial}{\partial\xi_i}\big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\big\rangle - \frac{\partial}{\partial\overline{\xi}_i}\big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\frac{\partial}{\partial\xi_i}\big\rangle\Big) \\
& + \frac{1}{12}\Big(\frac{\partial}{\partial\xi_i}\big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\big\rangle - \frac{\partial}{\partial\overline{\xi}_i}\big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\frac{\partial}{\partial\xi_i}\big\rangle\Big).
\end{aligned}
\tag{3.6.37}
$$

Using (3.2.25) we find that

$$
\begin{aligned}
& \frac{\partial}{\partial\xi_i}\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle - \frac{\partial}{\partial\overline{\xi}_i}\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\frac{\partial}{\partial\xi_i}\Big\rangle \\
=& 4\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle + \Big\langle 4\pi R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R} + R^{TX}(\mathcal{J}\mathcal{R},\mathcal{R})\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle.
\end{aligned}
\tag{3.6.38}
$$

Clearly,

$$
\begin{aligned}
& \frac{\partial}{\partial\xi_i}\Big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle - \frac{\partial}{\partial\overline{\xi}_i}\Big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\frac{\partial}{\partial\xi_i}\Big\rangle \\
=& 3\Big\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle + 4\pi\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle.
\end{aligned}
\tag{3.6.39}
$$

Substituting (3.6.38) and (3.6.39) into (3.6.37), we obtain

$$
\begin{aligned}
\mathcal{O}'_{2,1} =& \frac{1}{2}b_i\Big[\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i})\Big] \\
& + \frac{4\pi}{3}\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle + \Big\langle(\nabla^X\nabla^X\mathcal{J})_{\mathcal{R},\mathcal{R}}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle.
\end{aligned}
\tag{3.6.40}
$$

Combining (3.6.34), (3.6.35) and (3.6.40), we get

$$\begin{aligned}
\mathcal{O}_2'\mathcal{P} =&\mathcal{O}_{2,1}'\mathcal{P}+\mathcal{O}_{2,2}'\mathcal{P}-\frac{1}{3}\Big[\mathcal{L}_0,\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i}\rangle\Big]\mathcal{P} \\
&+R^E_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})b_i\mathcal{P}+\frac{1}{9}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2\mathcal{P}-\frac{4}{3}\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle\mathcal{P} \\
=&\frac{1}{2}b_i\Big[\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\Big]\mathcal{P}+\Big\langle(\nabla^X\nabla^X\mathcal{J})_{\mathcal{R},\mathcal{R}}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle\mathcal{P} \\
&+\Big\{\frac{1}{3}b_ib_j\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\rangle+4\langle R^{TX}_{x_0}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j})\frac{\partial}{\partial\bar{\xi}_i},\frac{\partial}{\partial\bar{\xi}_j}\rangle \\
&+\frac{4}{3}b_j\Big[\langle R^{TX}_{x_0}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_j}\rangle-\langle R^{TX}_{x_0}(\mathcal{R},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\bar{\xi}_i},\frac{\partial}{\partial\bar{\xi}_j}\rangle\Big]\Big\}\mathcal{P} \\
&-\frac{1}{3}\Big[\mathcal{L}_0,\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i}\rangle\Big]\mathcal{P}+R^E_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar{\xi}_i})b_i\mathcal{P}+\frac{1}{9}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2\mathcal{P}.
\end{aligned} \tag{3.6.41}$$

Now (3.6.30) follows immediately from (3.6.41). The proof of Lemma 3.38 is complete. □

Let $h(\xi)$ and $f(\xi)$ be arbitrary polynomials in ξ. From (3.4.210) and Theorem 3.31, we have

$$\begin{aligned}
\big(b_ih\mathcal{P}\big)(0,0)=-2\frac{\partial h}{\partial\xi_i}(0),\ \ \big(b_ib_jf\mathcal{P}\big)(0,0)=4\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(0), \\
\big(\mathcal{L}_0^{-1}b_ifb_j\mathcal{P}\big)(0,0)=-\frac{1}{2\pi}\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(0).
\end{aligned} \tag{3.6.42}$$

The following result is due to Ma and Marinescu, see [31, (2.39)].

Lemma 3.39.

$$-\Big(\mathcal{L}_0^{-1}\mathcal{P}^\perp\mathcal{O}_2'\mathcal{P}\Big)(0,0)=\frac{1}{2\pi}\Big[R^E(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i})+\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_j})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle\Big]. \tag{3.6.43}$$

Proof. We give the proof for the sake of completeness. If $U,V\in TX$, then

$$\Big\langle[R^{TX}(U,V),\mathbf{J}]\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_j}\Big\rangle=0. \tag{3.6.44}$$

In view of (3.2.25) and (3.6.44), we find that

$$\Big\langle\big(\nabla^X\nabla^X\mathbf{J}\big)_{(\xi,\bar{\xi})}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle\mathcal{P}(Z,Z')=\Big\langle\big(\nabla^X\nabla^X\mathbf{J}\big)_{(\bar{\xi},\xi)}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\bar{\xi}_i}\Big\rangle\mathcal{P}(Z,Z'). \tag{3.6.45}$$

By (3.6.6), (3.6.24) and (3.6.45), we get

$$\begin{aligned} & -\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\mathcal{P}\Big)(0,0) \\ =& -\frac{1}{2\pi}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\frac{\partial}{\partial\xi_i},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle \\ =& -\frac{1}{16\pi}\big|\nabla^X\mathbf{J}\big|^2. \end{aligned} \tag{3.6.46}$$

From (3.6.2) and Theorem 3.31, we find that

$$\begin{aligned} & \frac{1}{3}\Big(\mathcal{P}^{\perp}\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_i})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\mathcal{P}\Big)(0,0) \\ =& \frac{1}{3}\Big(\mathcal{P}^{\perp}\Big\langle R^{TX}(\xi,\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\overline{\xi}_j}+R^{TX}(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\xi_i})\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\frac{b_j}{2\pi}\mathcal{P}\Big)(0,0) \\ =& -\frac{1}{3\pi}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\overline{\xi}_j}+R^{TX}(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle \\ =& \frac{r^X}{24\pi}. \end{aligned} \tag{3.6.47}$$

It is a consequence of (3.2.29) that

$$-\frac{1}{9}\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2\mathcal{P}\Big)(0,0)=-\frac{8\pi^2}{9}\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}}\mathbf{J})\overline{\xi}\Big\rangle\mathcal{P}\Big)(0,0). \tag{3.6.48}$$

By (3.4.210), we deduce that

$$\begin{aligned} & \Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}}\mathbf{J})\overline{\xi}\Big\rangle\mathcal{P}(Z,Z') \\ =& \Big\{\Big[\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\frac{b_ib_j}{4\pi^2}+\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}'}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\frac{b_j}{2\pi} \\ & +\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\overline{\xi}'\Big\rangle\frac{b_i}{2\pi}+\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}'}\mathbf{J})\overline{\xi}'\Big\rangle\Big]\mathcal{P}\Big\}(Z,Z') \\ =& \Big\{\Big[\Big\langle\frac{b_ib_j}{4\pi^2}(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}+\frac{b_i}{2\pi}\Big((\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\overline{\xi}'+(\nabla^X_{\overline{\xi}'}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big),(\nabla^X_{\xi}\mathbf{J})\xi\Big\rangle \\ & +\frac{b_i}{2\pi^2}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\xi+(\nabla^X_{\xi}\mathbf{J})\frac{\partial}{\partial\xi_j},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}+(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle \\ & +\frac{1}{\pi}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_i}}\mathbf{J})\xi+(\nabla^X_{\xi}\mathbf{J})\frac{\partial}{\partial\xi_i},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\overline{\xi}'+(\nabla^X_{\overline{\xi}'}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}'}\mathbf{J})\overline{\xi}'\Big\rangle \\ & +\frac{1}{\pi^2}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_i}}\mathbf{J})\frac{\partial}{\partial\xi_j}+(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\frac{\partial}{\partial\xi_i},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\Big]\mathcal{P}\Big\}(Z,Z'). \end{aligned} \tag{3.6.49}$$

Using (3.6.10), (3.6.12) and Theorem 3.31, we find that

$$\begin{aligned}
&\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}}\mathbf{J})\overline{\xi}\Big\rangle\mathcal{P}\Big)(0,0)\\
=&\Big\{\Big[\frac{b_ib_j}{32\pi^3}\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\\
&+\frac{b_i}{8\pi^3}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\xi+(\nabla^X_{\xi}\mathbf{J})\frac{\partial}{\partial\xi_j},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}+(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\Big]\mathcal{P}\Big\}(0,0)\\
=&\frac{1}{8\pi^3}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_i}}\mathbf{J})\frac{\partial}{\partial\xi_j}+(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\frac{\partial}{\partial\xi_i},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle \qquad (3.6.50)\\
&-\frac{1}{4\pi^3}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\frac{\partial}{\partial\xi_i}+(\nabla^X_{\frac{\partial}{\partial\xi_i}}\mathbf{J})\frac{\partial}{\partial\xi_j},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}+(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\\
=&-\frac{3}{8\pi^3}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\frac{\partial}{\partial\xi_i}+(\nabla^X_{\frac{\partial}{\partial\xi_i}}\mathbf{J})\frac{\partial}{\partial\xi_j},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle=-\frac{9}{128\pi^3}\big|\nabla^X\mathbf{J}\big|^2.
\end{aligned}$$

Now (3.6.48) and (3.6.50) imply

$$-\frac{1}{9}\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\Big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\Big|^2\mathcal{P}\Big)(0,0)=\frac{1}{16\pi}\big|\nabla^X\mathbf{J}\big|^2. \qquad (3.6.51)$$

In view of (3.4.210), when calculating $-\big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\mathcal{O}_2'\mathcal{P}\big)(0,0)$, the contribution of the term $\frac{1}{2}b_i\sum_{|\alpha|=2}(\partial^{\alpha}R^L)_{x_0}(\mathcal{R},\frac{\partial}{\partial\overline{\xi}_i})\frac{Z^{\alpha}}{\alpha!}$ in $\mathcal{O}_2'$ consists of the terms whose total degree of b_i and $\overline{\xi}_j$ is the same as the degree of ξ. Hence we only consider the contribution from the terms where the degree of ξ is 2. From [31, (2.7)], the contribution is

$$\begin{aligned}
I_2=&\frac{1}{4}b_i\Big[\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(\xi,\xi)}\overline{\xi}+(\nabla^X\nabla^X\mathcal{J})_{(\xi,\overline{\xi})}\xi+(\nabla^X\nabla^X\mathcal{J})_{(\overline{\xi},\xi)}\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle \qquad (3.6.52)\\
&+\frac{1}{3}\Big\langle R^{TX}(\overline{\xi},\mathcal{J}\xi)\xi+R^{TX}(\xi,\mathcal{J}\overline{\xi})\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\Big].
\end{aligned}$$

By (3.2.25), (3.2.26) and (3.2.29), we get

$$\Big\langle(\nabla^X\nabla^X\mathbf{J})_{(\xi,\xi)}\overline{\xi},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle=\Big\langle(\nabla^X\nabla^X\mathbf{J})_{(\xi,\overline{\xi})}\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\frac{\sqrt{-1}}{2}\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_i}}\mathbf{J})\overline{\xi}\Big\rangle, \quad (3.6.53)$$

and

$$\begin{aligned}
\Big\langle(\nabla^X\nabla^X\mathbf{J})_{(\xi,\overline{\xi})}\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle&=-\frac{\sqrt{-1}}{2}\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle, \qquad (3.6.54)\\
\Big\langle(\nabla^X\nabla^X\mathbf{J})_{(\overline{\xi},\xi)}\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle&=-\frac{\sqrt{-1}}{2}\Big\langle(\nabla^X_{\xi}\mathbf{J})\xi,(\nabla^X_{\overline{\xi}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle.
\end{aligned}$$

Using (3.6.42), we find that

$$\begin{aligned}
&\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}b_i\Big\langle R^{TX}(\overline{\xi},\xi)\xi,\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\mathcal{P}\Big)(0,0) \qquad (3.6.55)\\
=&-\frac{1}{4\pi^2}\Big(\Big\langle R^{TX}(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\Big\langle R^{TX}(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\xi_j})\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\Big)\\
=&-\frac{1}{4\pi^2}\Big(2\Big\langle R^{TX}(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j})\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\Big).
\end{aligned}$$

From (3.6.10), (3.6.12), (3.6.13), (3.6.42) and (3.6.52)–(3.6.55), we obtain

$$\begin{aligned}
&-\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}I_2\,\mathcal{P}\Big)(0,0)\\
=&-\frac{\pi}{3}\Big[\frac{1}{2\pi^2}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_j})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\frac{1}{4\pi^2}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j})\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\Big]\\
&-\frac{1}{8\pi}\Big\langle(\nabla^X_{\frac{\partial}{\partial\xi_i}}\mathbf{J})\frac{\partial}{\partial\xi_j}+(\nabla^X_{\frac{\partial}{\partial\xi_j}}\mathbf{J})\frac{\partial}{\partial\xi_i},(\nabla^X_{\frac{\partial}{\partial\overline{\xi}_j}}\mathbf{J})\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle \qquad (3.6.56)\\
=&-\frac{1}{6\pi}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_j})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle-\frac{1}{12\pi}\cdot\frac{5}{16}\big|\nabla^X\mathbf{J}\big|^2.
\end{aligned}$$

Combining (3.6.30), (3.6.46), (3.6.47), (3.6.51) and (3.6.56), we get

$$\begin{aligned}
&-\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\mathcal{O}_2'\mathcal{P}\Big)(0,0)\\
=&-\Big\{\Big[\frac{b_ib_j}{24\pi}\Big\langle R^{TX}(\xi,\frac{\partial}{\partial\overline{\xi}_i})\xi,\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle+\frac{b_i}{4\pi}R^E(\xi,\frac{\partial}{\partial\overline{\xi}_i})\\
&+\frac{b_j}{3\pi}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\xi-R^{TX}(\xi,\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\Big]\mathcal{P}\Big\}(0,0)\\
&+\frac{1}{3}\Big(\mathcal{P}^{\perp}\Big\langle R^{TX}(\mathcal{R},\frac{\partial}{\partial\xi_j})\mathcal{R},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\mathcal{P}\Big)(0,0)-\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}I_2\,\mathcal{P}\Big)(0,0)\\
&-\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle\mathcal{P}\Big)(0,0)\\
&-\frac{1}{9}\Big(\mathcal{L}_0^{-1}\mathcal{P}^{\perp}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2\mathcal{P}\Big)(0,0) \qquad (3.6.57)\\
=&-\frac{1}{6\pi}\Big(\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle+\Big\langle R^{TX}(\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\Big)\\
&+\frac{1}{2\pi}R^E(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})+\frac{2}{3\pi}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\xi_j}-R^{TX}(\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_i})\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\\
&+\frac{r^X}{24\pi}-\frac{1}{6\pi}\Big[\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_j})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\frac{5}{32}\big|\nabla^X\mathbf{J}\big|^2\Big]\\
=&\frac{1}{6\pi}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_j})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_i}\Big\rangle+\frac{7}{6\pi}\Big\langle R^{TX}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j})\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\\
&+\frac{1}{2\pi}R^E(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})+\frac{r^X}{24\pi}-\frac{5}{192\pi}\big|\nabla^X\mathbf{J}\big|^2.
\end{aligned}$$

Then (3.6.1), (3.6.13) and (3.6.57) entail (3.6.43). The proof of Lemma 3.39 is complete. □

We now compute the term $\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\Psi P^N\Big)(0,0)$.

Note $\Psi = \frac{1}{4}(dT_{as}) \in \Lambda^{2,2}$. By (3.3.1),

$$\begin{aligned}
24\Psi =& \frac{1}{4}\big(dT_{as}\big)(e_i,e_j,e_k,e_k)c(e_i)c(e_j)c(e_k)c(e_l)\\
=& -\frac{1}{2}\big(dT_{as}\big)(e_i,e_j,v_k,\overline{v}_k)c(e_i)c(e_j) + \big(dT_{as}\big)(e_i,e_j,v_k,\overline{v}_l)c(e_i)c(e_j)\overline{v}^l\wedge i_{\overline{v}_k}\\
&+\frac{1}{2}\big(dT_{as}\big)(e_i,e_j,v_k,v_l)c(e_i)c(e_j)i_{\overline{v}_k}i_{\overline{v}_l}\\
&+\frac{1}{2}\big(dT_{as}\big)(e_i,e_j,\overline{v}_k,\overline{v}_l)c(e_i)c(e_j)\overline{v}^k\wedge\overline{v}^l\\
=& -2\Big[-\frac{1}{2}\big(dT_{as}\big)(v_j,\overline{v}_j,v_k,\overline{v}_k) + \big(dT_{as}\big)(v_i,\overline{v}_j,v_k,\overline{v}_k)\overline{v}^j\wedge i_{\overline{v}_i}\Big]\\
&+4\Big[-\frac{1}{2}\big(dT_{as}\big)(v_j,\overline{v}_j,v_k,\overline{v}_l) + \big(dT_{as}\big)(v_i,\overline{v}_j,v_k,\overline{v}_l)\overline{v}^j\wedge i_{\overline{v}_i}\Big]\overline{v}^l\wedge i_{\overline{v}_k}\\
&+\big(dT_{as}\big)(\overline{v}_i,\overline{v}_j,v_k,v_l)\overline{v}^i\wedge\overline{v}^j\wedge i_{\overline{v}_k}i_{\overline{v}_l}\\
&+\big(dT_{as}\big)(v_i,v_j,\overline{v}_k,\overline{v}_l)i_{\overline{v}_i}i_{\overline{v}_j}\overline{v}^k\wedge\overline{v}^l\\
=&\big(dT_{as}\big)(v_j,\overline{v}_j,v_k,\overline{v}_k) - 4\big(dT_{as}\big)(v_i,\overline{v}_j,v_k,\overline{v}_k)\overline{v}^j\wedge i_{\overline{v}_i}\\
&+4\big(dT_{as}\big)(v_i,\overline{v}_j,v_k,\overline{v}_l)\overline{v}^j\wedge i_{\overline{v}_i}\overline{v}^l\wedge i_{\overline{v}_k}\\
&+\big(dT_{as}\big)(\overline{v}_i,\overline{v}_j,v_k,v_l)\overline{v}^i\wedge\overline{v}^j\wedge i_{\overline{v}_k}i_{\overline{v}_l}\\
&+\big(dT_{as}\big)(v_i,v_j,\overline{v}_k,\overline{v}_l)i_{\overline{v}_i}i_{\overline{v}_j}\overline{v}^k\wedge\overline{v}^l.
\end{aligned}$$

Using the relation

$$\overline{v}^i\wedge i_{\overline{v}_j} + i_{\overline{v}_j}\overline{v}^i = \delta_{ij}, \tag{3.6.58}$$

we get

$$\begin{aligned}
24\Psi =& 3\big(dT_{as}\big)(v_i,\overline{v}_i,v_j,\overline{v}_j) - 12\big(dT_{as}\big)(v_i,\overline{v}_j,v_k,\overline{v}_k)\overline{v}^j\wedge i_{\overline{v}_i}\\
&+6\big(dT_{as}\big)(v_i,v_j,\overline{v}_k,\overline{v}_l)\overline{v}^k\wedge\overline{v}^l\wedge i_{\overline{v}_i}i_{\overline{v}_j}.
\end{aligned}$$

That is

$$\begin{aligned}
\Psi =& \frac{1}{2}\big(dT_{as}\big)\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_j}\Big) - 2\big(dT_{as}\big)\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial z_k},\frac{\partial}{\partial \overline{z}_k}\Big)d\overline{z}_j\wedge i_{\frac{\partial}{\partial \overline{z}_i}}\\
&+\big(dT_{as}\big)\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial \overline{z}_l}\Big)d\overline{z}_k\wedge d\overline{z}_l\wedge i_{\frac{\partial}{\partial \overline{z}_i}}i_{\frac{\partial}{\partial \overline{z}_j}}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\Psi I_{\det(\overline{W}^*)\otimes E} =&\frac{1}{2}\big(dT_{as}\big)(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_j})I_{\det(\overline{W}^*)\otimes E}\\
&-2\sum_{i=1}^{q}\big(dT_{as}\big)(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial z_k},\frac{\partial}{\partial \overline{z}_k})I_{\det(\overline{W}^*)\otimes E}\\
&-2\sum_{i=1}^{q}\sum_{j=q+1}^{n}\big(dT_{as}\big)(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial z_k},\frac{\partial}{\partial \overline{z}_k})d\overline{z}_j\wedge i_{\frac{\partial}{\partial \overline{z}_i}}I_{\det(\overline{W}^*)\otimes E}\\
&+2\sum_{i,j=1}^{q}\big(dT_{as}\big)(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_i})I_{\det(\overline{W}^*)\otimes E}\\
&+4\sum_{i,j=1}^{q}\sum_{k=q+1}^{n}\big(dT_{as}\big)(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial \overline{z}_i})d\overline{z}_k\wedge i_{\frac{\partial}{\partial \overline{z}_j}}I_{\det(\overline{W}^*)\otimes E}\\
&+\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}\big(dT_{as}\big)(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial \overline{z}_l})d\overline{z}_k\wedge d\overline{z}_l\wedge i_{\frac{\partial}{\partial \overline{z}_i}}i_{\frac{\partial}{\partial \overline{z}_j}}I_{\det(\overline{W}^*)\otimes E}\\
=&\frac{1}{2}\big(dT_{as}\big)(\frac{\partial}{\partial \xi_i},\frac{\partial}{\partial \overline{\xi}_i},\frac{\partial}{\partial \xi_j},\frac{\partial}{\partial \overline{\xi}_j})I_{\det(\overline{W}^*)\otimes E}\\
&-2\sum_{i=1}^{q}\sum_{j=q+1}^{n}\big(dT_{as}\big)(\frac{\partial}{\partial \overline{\xi}_i},\frac{\partial}{\partial \overline{\xi}_j},\frac{\partial}{\partial \xi_k},\frac{\partial}{\partial \overline{\xi}_k})d\overline{\xi}_j\wedge i_{\frac{\partial}{\partial \overline{\xi}_i}}I_{\det(\overline{W}^*)\otimes E}\\
&+\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}\big(dT_{as}\big)(\frac{\partial}{\partial \overline{\xi}_i},\frac{\partial}{\partial \overline{\xi}_j},\frac{\partial}{\partial \overline{\xi}_k},\frac{\partial}{\partial \overline{\xi}_l})d\overline{\xi}_k\wedge d\overline{\xi}_l\wedge i_{\frac{\partial}{\partial \overline{\xi}_i}}i_{\frac{\partial}{\partial \overline{\xi}_j}}I_{\det(\overline{W}^*)\otimes E}.
\end{aligned}
$$

Now it follows immediately that

$$
\begin{aligned}
&\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\Psi P^N\Big)(0,0) \qquad (3.6.59)\\
=&\frac{1}{16\pi}\Big[\sum_{i,j=1}^{q}\sum_{k,l=q+1}^{n}(dT_{as})(\frac{\partial}{\partial \overline{\xi}_i},\frac{\partial}{\partial \overline{\xi}_j},\frac{\partial}{\partial \overline{\xi}_k},\frac{\partial}{\partial \overline{\xi}_l})d\overline{\xi}_k\wedge d\overline{\xi}_l\wedge i_{\frac{\partial}{\partial \overline{\xi}_i}}i_{\frac{\partial}{\partial \overline{\xi}_j}}\\
&-\frac{\sqrt{-1}}{2}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\big(\Lambda_w(dT_{as})\big)(\frac{\partial}{\partial \overline{\xi}_j},\frac{\partial}{\partial \overline{\xi}_k})d\overline{\xi}_k\wedge i_{\frac{\partial}{\partial \overline{\xi}_j}}\Big]I_{\det(\overline{W}^*)\otimes E}.
\end{aligned}
$$

By (3.5.4), we get

$$
\begin{aligned}
\mathcal{O}_2 =&\mathcal{O}_2'+R_{x_0}^{B,\Lambda^{0,\bullet}}(\mathcal{R},\frac{\partial}{\partial \overline{\xi}_j})b_j-R_{x_0}^{B,\Lambda^{0,\bullet}}(\mathcal{R},\frac{\partial}{\partial \xi_j})b_j^+ \qquad (3.6.60)\\
&-\frac{\sqrt{-1}}{2}\pi\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\mathcal{R},\mathcal{R})}e_l,e_m\Big\rangle c(e_l)c(e_m)\\
&+\frac{1}{2}\Big(R_{x_0}^E+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)(e_l,e_m)c(e_l)c(e_m)+\frac{1}{4}r_{x_0}^X-\Psi.
\end{aligned}
$$

By (3.3.1) and (3.6.15),

$$\begin{aligned} & P^{N^\perp}\big(\mathcal{O}_2 - \mathcal{O}_2' + \Psi\big)P^N \qquad (3.6.61)\\ =& P^{N^\perp}\Big\{\frac{1}{2}\mathrm{Tr}[R^{T^{(1,0)}X}]\big(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i}\big)b_i \\ & - \Big\langle\Big(R^B(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})b_i - 2\pi\sqrt{-1}(\nabla^B\nabla^B\mathbf{J})_{(\mathcal{R},\mathcal{R})}\Big)\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle \\ & + 2\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big\{\Big\langle\big(R^B(\mathcal{R}, \frac{\partial}{\partial \overline{\xi}_i})b_i - 2\sqrt{-1}\pi(\nabla^B\nabla^B\mathbf{J})_{(\mathcal{R},\mathcal{R})}\big)\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\Big\rangle \\ & \qquad + 2\Big(R^E + \frac{1}{2}\mathrm{Tr}[R^{T^{(1,0)}X}]\Big)\big(\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\big)\Big\}d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}}\Big\}P^N. \end{aligned}$$

Using (3.2.25), (3.4.210), (3.6.24)–(3.6.25) and Theorem 3.31, we have

$$\begin{aligned} & -\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\big(\mathcal{O}_2 - \mathcal{O}_2' + \Psi\big)P^N\Big)(0,0) \qquad (3.6.62)\\ =& \Big\{\frac{1}{4\pi}\mathrm{Tr}[R^{T^{(1,0)}X}]\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big) - \frac{1}{2\pi}\Big\langle R^B\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big)\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle \\ & + \frac{\sqrt{-1}}{2\pi}\Big\langle\Big(2(\nabla^B\nabla^B\mathbf{J})_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})} - \big[R^B\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big), \mathbf{J}\big]\Big)\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle \\ & - \sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big\{\frac{1}{6\pi}\Big\langle R^B_{x_0}\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big)\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\Big\rangle + \frac{1}{2\pi}\Big(R^E + \frac{1}{2}\mathrm{Tr}[R^{T^{(1,0)}X}]\Big)\big(\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\big) \\ & - \frac{\sqrt{-1}}{6\pi}\Big\langle\Big(2\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})} - \big[R^B\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big), \mathbf{J}\big]\Big)\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\Big\rangle\Big\}d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}}\Big\}I_{\det(\overline{W}^*)\otimes E}. \end{aligned}$$

Using (3.6.44), we get

$$\begin{aligned} & -\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\big(\mathcal{O}_2 - \mathcal{O}_2' + \Psi\big)P^N\Big)(0,0) \qquad (3.6.63)\\ =& \Big\{\frac{1}{4\pi}\mathrm{Tr}[R^{T^{(1,0)}X}]\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big) - \frac{1}{2\pi}\sum_{j=1}^{n}\Big\langle R^B_{x_0}\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big)\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle \\ & + \frac{\sqrt{-1}}{\pi}\sum_{j=1}^{n}\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})}\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \overline{\xi}_j}\Big\rangle \\ & - \frac{1}{2\pi}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big[\Big\langle R^B_{x_0}\big(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i}\big)\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\Big\rangle \\ & \qquad - \frac{2\sqrt{-1}}{3}\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_i})}\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\Big\rangle \\ & \qquad + \Big(R^E + \frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)\big(\frac{\partial}{\partial \overline{\xi}_j}, \frac{\partial}{\partial \overline{\xi}_k}\big)\Big]d\overline{\xi}_k \wedge i_{\frac{\partial}{\partial \overline{\xi}_j}}\Big\}I_{\det(\overline{W}^*)\otimes E}. \end{aligned}$$

By (3.2.28), we obtain

$$\sqrt{-1}\sum_{j=1}^{n}\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})}\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle=\frac{1}{16}\big|\nabla^B\mathbf{J}\big|^2. \tag{3.6.64}$$

Substituting (3.6.64) into (3.6.63), we obtain

$$\begin{aligned}
&-\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\big(\mathcal{O}_2-\mathcal{O}_2'+\Psi\big)P^N\Big)(0,0)\\
=&\Big[\frac{1}{16\pi}\big|\nabla^B\mathbf{J}\big|^2-\frac{1}{2\pi}\Big\langle R^B(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\\
&+\frac{1}{4\pi}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\Big]I_{\det(\overline{W}^*)\otimes E}\\
&-\frac{1}{2\pi}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big[\Big\langle R^B_{x_0}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k}\Big\rangle\\
&\qquad-\frac{2\sqrt{-1}}{3}\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})}\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k}\Big\rangle\\
&\qquad+\Big(R^E+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k})\Big]d\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\xi_j}}I_{\det(\overline{W}^*)\otimes E}.
\end{aligned}\tag{3.6.65}$$

Combining (3.6.13), (3.6.43) and (3.6.65) together, we finally get

$$\begin{aligned}
&-\Big((\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N\Big)(0,0)\\
=&\Big[\frac{1}{16\pi}\big|\nabla^B\mathbf{J}\big|^2-\frac{1}{64\pi}\big|\nabla^X\mathbf{J}\big|^2+\frac{1}{2\pi}R^E(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})+\frac{1}{4\pi}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\\
&-\frac{1}{2\pi}\Big\langle\big(R^B-R^{TX}\big)(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_j}\Big\rangle\Big]I_{\det(\overline{W}^*)\otimes E}\\
&-\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big[\frac{1}{2\pi}\Big\langle R^B_{x_0}(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k}\Big\rangle+\frac{1}{4\pi}(dT_{as})(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k})\\
&\qquad-\frac{\sqrt{-1}}{3\pi}\Big\langle\big(\nabla^B\nabla^B\mathbf{J}\big)_{(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})}\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k}\Big\rangle\\
&\qquad+\frac{1}{2\pi}\Big(R^E+\frac{1}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k})\Big]d\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\xi_j}}\\
&+\frac{1}{16\pi}(dT_{as})(\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\overline{\xi}_k},\frac{\partial}{\partial\overline{\xi}_l})d\overline{\xi}_k\wedge d\overline{\xi}_l\wedge i_{\frac{\partial}{\partial\xi_i}}i_{\frac{\partial}{\partial\xi_j}}I_{\det(\overline{W}^*)\otimes E}.
\end{aligned}\tag{3.6.66}$$

By (3.2.35), we obtain

$$
\begin{aligned}
&-\Big((\mathcal{L}_2^0)^{-1}\mathcal{P}^{N^\perp}\mathcal{O}_2\mathcal{P}^N\Big)(0,0)\\
=&\Big[\frac{1}{2\pi}R^E(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})+\frac{1}{4\pi}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big](\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})-\frac{1}{32\pi}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)\\
&+\frac{3}{64\pi}\big|\nabla^B\mathbf{J}\big|^2+\frac{2}{\pi}\Big|\big\langle S^B(\frac{\partial}{\partial\overline{\xi}_i})\frac{\partial}{\partial\xi_j},\frac{\partial}{\partial\xi_k}\big\rangle\Big|^2\Big]I_{\det(\overline{W}^*)\otimes E}\\
&-\frac{1}{2\pi}\sum_{j=1}^{q}\sum_{k=q+1}^{n}\Big[P(\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k})\\
&\qquad-\frac{2\sqrt{-1}}{3}\big\langle(\nabla^B\nabla^B\mathbf{J})_{(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\overline{\xi}_i})}\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k}\big\rangle\Big]d\overline{\xi}_k\wedge i_{\frac{\partial}{\partial\overline{\xi}_j}}I_{\det(\overline{W}^*)\otimes E}\\
&+\frac{1}{16\pi}(dT_{as})(\frac{\partial}{\partial\overline{\xi}_i},\frac{\partial}{\partial\overline{\xi}_j},\frac{\partial}{\partial\overline{\xi}_k},\frac{\partial}{\partial\overline{\xi}_l})d\overline{\xi}_k\wedge d\overline{\xi}_l\wedge i_{\frac{\partial}{\partial\overline{\xi}_i}}i_{\frac{\partial}{\partial\overline{\xi}_j}}I_{\det(\overline{W}^*)\otimes E}.
\end{aligned}
\tag{3.6.67}
$$

Now our main result (3.1.19) follows immediately from (3.5.28), (3.5.29), (3.6.22), (3.6.23), (3.6.27) and (3.6.67). This completes the proof of Theorem 3.3.

Proof of Corollary 3.4. Since (X, g^{TX}, J) is Kähler, then the torsion T vanishes, hence

$$T_{as}=0,\ \nabla^B=\nabla^{TX},\ \nabla^B\mathbf{J}=\nabla^X\mathbf{J},\ \text{and}\ R^B=R^{TX}. \tag{3.6.68}$$

From (3.2.25) and (3.2.29), we know

$$\Big\langle\big(\nabla^X\nabla^X\mathbf{J}\big)_{(\overline{u}_i,\overline{u}_j)}\overline{u}_k,u_l\Big\rangle=\Big\langle\big(\nabla^X\nabla^X\mathbf{J}\big)_{(\overline{u}_i,\overline{u}_k)}u_l,\overline{u}_j\Big\rangle=0. \tag{3.6.69}$$

By (3.2.26) and (3.6.69), we obtain

$$\Big\langle\big(\nabla^X\nabla^X\mathbf{J}\big)_{(\overline{u}_i,u_i)}\overline{u}_j,\overline{u}_k\Big\rangle=0, \tag{3.6.70}$$

which implies

$$\Big\langle\big(\nabla^X\nabla^X\mathbf{J}\big)_{(u_i,\overline{u}_i)}\overline{u}_j,\overline{u}_k\Big\rangle=-2\sqrt{-1}\Big\langle R^{TX}(u_i,\overline{u}_i)\overline{u}_j,\overline{u}_k\Big\rangle. \tag{3.6.71}$$

Formula (3.1.21) follows from (3.1.19), (3.2.29), (3.6.68) and (3.6.71). The proof of Corollary 3.4 is complete. □

3.7 Compatibility with Riemann-Roch-Hirzebruch formula

In this Section we check the compatibility of our final result (3.1.19) with Riemann-Roch-Hirzebruch formulas.

Let $h_p^{0,q}$ be the dimension of $H^{0,q}(X, L^p \otimes E)$, and let $\mathrm{rk}(E)$ be the rank of E. Combining (3.1.8) and the Riemann-Roch-Hirzebruch Theorem (cf. e.g. [30, Theorem 1.4.6]), we find that

$$\begin{aligned}(-1)^q h_p^{0,q} &= \int_X \mathrm{Td}\big(T^{(1,0)}X\big)\mathrm{ch}(L^p \otimes E) \\ &= \mathrm{rk}(E)\int_X \frac{c_1(L)^n}{n!}p^n + \int_X \Big(c_1(E) + \frac{\mathrm{rk}(E)}{2}c_1\big(T^{(1,0)}X\big)\Big)\frac{c_1(L)^{n-1}}{(n-1)!}p^{n-1} \\ &\quad + O(p^{n-2}),\end{aligned} \tag{3.7.1}$$

where $\mathrm{ch}(\cdot), c_1(\cdot), \mathrm{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles, respectively.

By integrating over X the expansion (3.1.11) for $k = 1$, we have

$$\begin{aligned}&\int_X \mathrm{Tr}\big[P_p^{0,q}(x,x)\big]dv_X(x) \\ =&p^n \int_X \mathrm{Tr}\big[\mathbf{b}_0(x)\big]dv_X(x) + p^{n-1}\int_X \mathrm{Tr}\big[\mathbf{b}_1(x)\big]dv_X(x) + O(p^{n-2}),\end{aligned} \tag{3.7.2}$$

where the trace is taken over $\Lambda^q(T^{*(0,1)}X) \otimes E$. By (3.1.9) and (3.1.13), we obtain

$$dv_X = \Theta^n/n! = (-1)^q\omega^n/n!. \tag{3.7.3}$$

It follows from (3.1.15) that the following identity holds for any smooth 2-form α,

$$\alpha \wedge \omega^{n-1}/(n-1)! = -\sqrt{-1}\alpha(u_j, \overline{u}_j)\cdot \omega^n/n! = (\Lambda_\omega \alpha)\omega^n/n!. \tag{3.7.4}$$

Applying (3.7.4) for $\alpha = d(\Lambda_\omega T_{as})$ and the Stokes' Theorem, we obtain

$$\int_X \Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)dv_X = (-1)^q/(n-1)!\cdot \int_X d(\Lambda_\omega T_{as})\wedge\omega^{n-1} = 0. \tag{3.7.5}$$

Substituting (3.1.20), (3.7.3), (3.7.5) and the equality (3.7.4) for $\alpha = c_1(E)$ and $c_1(T^{(1,0)}X)$, respectively, into (3.7.2), we obtain (3.7.1). Therefore, our final formula (3.1.19) is compatible with (3.7.1).

On the other hand we also explain here the compatibility of our formula (3.1.19) with the local index formula obtained by Bismut [5, (2.53)] for non Kähler manifolds under the assumption that the form T_{as} is closed.

Recall that S^B is defined in (3.2.14). Set

$$\nabla^{-B} = \nabla^{TX} + S^{-B} \quad \text{with } S^{-B} = -S^B. \tag{3.7.6}$$

We denote by R^{-B} the curvature of the connection ∇^{-B}. Note that by (3.2.14) and [5, (2.36)] our notations S^B, R^{-B} correspond to S^{-B} and R^B in [5, §II b)] respectively. Let $\widehat{A}$ be the Hirzebruch $\widehat{A}$-polynomial on $(2n, 2n)$ matrices. Then

$$\widehat{A}\Big(\frac{R^{-B}}{2\pi}\Big) \in \oplus_j\ \Omega^{4j}(X), \tag{3.7.7}$$

where $\Omega^j(X)$ denotes the space of smooth j-forms over X.

For $t > 0$, let $Q_{p,t}(x,y)$ be the smooth kernel on X associated to the operator $\exp(-tD_p^2)$. Let $\Omega^{0,\mathrm{even}}(X, L^p \otimes E)$ (resp. $\Omega^{0,\mathrm{odd}}(X, L^p \otimes E)$) be the direct sum of the space of smooth $(0,2j)$-forms (resp. $(0,2j+1)$-forms) over X with values in $L^p \otimes E$ for $j \geqslant 0$. Set

$$\mathrm{Tr}_s = \mathrm{Tr}\big|_{\Omega^{0,\mathrm{even}}(X,L^p\otimes E)} - \mathrm{Tr}\big|_{\Omega^{0,\mathrm{odd}}(X,L^p\otimes E)}. \tag{3.7.8}$$

Note that the auxiliary vector bundle ξ in [5, Theorem 2.11] should read as $L^p \otimes E$. Denote by $R^{L^p\otimes E}$ the curvature of the Chern connection $\nabla^{L^p\otimes E}$ on $L^p \otimes E$. Then we can restate [5, Theorem 2.11] as follows.

Theorem 3.40. *Assume that $dT_{as} = 0$, then*

$$\begin{aligned}&\lim_{t\to 0} \mathrm{Tr}_s\big[Q_{p,t}(x,x)\big]dv_X(x) \\ =&\Big\{\widehat{A}\big(\frac{R^{-B}}{2\pi}\big)\exp\Big(\frac{\sqrt{-1}}{4\pi}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)\mathrm{Tr}\Big[\exp\big(\frac{\sqrt{-1}}{2\pi}R^{L^p\otimes E}\big)\Big]\Big\}^{\max}\end{aligned} \tag{3.7.9}$$

uniformly on X.

Now we check the compatibility of our final result (3.1.19) with (3.7.9).

Mckean-Singer formula [2, Th. 3.50] also holds for the modified Dirac operator D_p:

$$\sum_{j=0}^{n}(-1)^j \ \dim H^{0,j}(X, L^p \otimes E) = \int_X \mathrm{Tr}_s\big[Q_{p,t}(x,x)\big]dv_X(x) \ \text{ for any } t > 0. \tag{3.7.10}$$

Combining (3.1.8), (3.7.9) and (3.7.10) yields

$$(-1)^q h_p^{0,q} = \int_X \widehat{A}\big(\frac{R^{-B}}{2\pi}\big)\exp\Big(\frac{\sqrt{-1}}{4\pi}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)\mathrm{Tr}\Big[\exp\big(\frac{\sqrt{-1}}{2\pi}R^{L^p\otimes E}\big)\Big]. \tag{3.7.11}$$

If we expand the right hand side of the formula (3.7.11) in a polynomial of degree n in p, then it follows from (3.7.7) that the term $\widehat{A}(\frac{R^{-B}}{2\pi})$ has no contribution to the coefficients of p^n and p^{n-1}. Hence we obtain from (3.7.11) that

$$\begin{aligned}(-1)^q h_p^{0,q} = \mathrm{rk}(E)\int_X \frac{\omega^n}{n!}p^n + \frac{\sqrt{-1}}{2\pi}\int_X \Big(\mathrm{Tr}[R^E] + \frac{\mathrm{rk}(E)}{2}\mathrm{Tr}\big[R^{T^{(1,0)}X}\big]\Big)\frac{\omega^{n-1}}{(n-1)!}p^{n-1} \\ + O(p^{n-2}),\end{aligned} \tag{3.7.12}$$

which can be obtained by replacing the cohomology classes in (3.7.1) by the corresponding Chern-Weil forms. We find that the coefficient of p^{n-1} in (3.7.12) coincides pointwise to (3.1.20) modulo the term $-\frac{1}{16}\Lambda_\omega\big(d(\Lambda_\omega T_{as})\big)$. This fits well the compatibility of the asymptotic expansion of Bergman kernel and the local index theorem along the lines of [30, Remark 4.1.4], [32, §5.1].

4 Appendix

In this Chapter, we give a complete proof of the following fact. Two basic references here are [35] and [45].

Theorem 4.1. *Given a Morse function on a smooth closed manifold, there always exists a Riemannian metric such that the minus gradient vector field of the function associated to the metric verifies the Morse-Smale conditions.*

Let M be a smooth closed oriented manifold and f denote a Morse function. Let g^{TM} be a Riemannian metric on TM. Denote by ∇f be the gradient vector filed of f associated to g^{TM}. Let $C(f)$ consist of the critical points of f. Set $X = -\nabla(f)$. Denote by φ_t be flow lines of X, i.e., for $x \in M$,

$$\frac{d\varphi_t(x)}{dt} = X_{\varphi_t(x)}, \;\; \varphi_0(x) = x. \tag{4.0.1}$$

If $p \in C(f)$ with index λ, the unstable (resp. stable) manifold $W^u(p)$ $\big($resp. $W^s(p)\big)$ in the X system is given by

$$W^u(p) = \big\{x \in M, | \lim_{t\to-\infty} \varphi_t(x) = p\big\} \tag{4.0.2}$$

$\Big($resp.

$$W^s(p) = \big\{x \in M, | \lim_{t\to\infty} \varphi_t(x) = p\big\}\Big). \tag{4.0.3}$$

Moreover,

$$\dim W^u(p) = \lambda; \;\; \dim W^s(p) = n - \lambda. \tag{4.0.4}$$

Definition 4.2. We say the vector field X satisfies the Morse-Smale conditions if for any $p, q \in C(f)$, $W^u(p)$ and $W^s(q)$ intersect transversally, which we usually denote by $W^u(p) \pitchfork W^s(q)$.

Definition 4.3. A vector field ξ on M is called a minus gradient-like vector field for f if

1). $\xi.f < 0$ over $M \backslash C(f)$;

2). given $p \in C(f)$ with index λ there are coordinates $x = (x_1, \cdots, x_n)$ in a neighborhood U of p so that

$$f = f(p) - \frac{x_1^2}{2} - \cdots - \frac{x_\lambda^2}{2} + \frac{x_{\lambda+1}^2}{2} + \cdots + \frac{x_n^2}{2}, \tag{4.0.5}$$

and

$$\xi = \sum_{j=1}^{\lambda} x_j \frac{\partial}{\partial x_j} - \sum_{j=\lambda+1}^{n} x_j \frac{\partial}{\partial x_j} \tag{4.0.6}$$

holds throughout U.

Clearly, $X = -\nabla f$ is a minus gradient-like vector field if g^{TM} is the Euclidean metric on U. Conversely, we have the following results.

Proposition 4.4. *Given ξ a minus gradient-like vector field, there always exists a Riemmannian metric such that ξ is exactly the minus gradient vector field of f with respect to the chosen metric.*

Proof. For $p \in C(f)$, let (V_p, ϕ_p) be coordinates system appeared Definition 4.3. We assume there exists $\delta_p > 0$ such that $B(0, 2\delta_p) \subset \phi_p(V_p)$, where $B(0, 2\delta_p)$ is the ball in $\mathbb{R}^n$ centered in 0 with radius $2\delta_p$. Set $U_p = \phi_p^{-1}\big(B(0, 2\delta_p)\big), U'_p = \phi_p^{-1}\big(B(0, \delta_p)\big)$. Then we define

$$g_0^{TM} = \sum_{j=1}^{n} dx_j \otimes dx_j, \qquad \text{on } B(0, 2\delta_p). \tag{4.0.7}$$

Set $M_1 = M - \cup_p \overline{U'_p}$. Then M_1 is an open submanifold of M.
For $x \in M_1$, set

$$N_x = \big\{X \in T_x M |\ (df)_x X = X_x f = 0\big\}. \tag{4.0.8}$$

That is, N_x is the kernel space of $(df)_x$. Since $df \neq 0$ on M_1, N is a $(n-1)$-dimensional subbundle of $TM|_{M_1}$. There is a Riemaninan metric g^N on the subbundle N. Since ξ does not belong to N, then we could define a metric g_1^{TM} on $TM|_{M_1} = \mathbb{R}\xi \oplus N$ such that $g_1^{TM} = g^\xi \oplus g^N$, here g^ξ is defined by:

$$g_x^\xi(\xi, \xi) = -\xi_x(f), \quad \forall\, x \in M_1. \tag{4.0.9}$$

Let ρ_0 and ρ_1 be the partition of unit associated to the open covering $\cup_p U_p$ and M_1. Set

$$g^{TM} = \rho_0 g_0^{TM} \oplus \rho_1 g_1^{TM}. \tag{4.0.10}$$

It is clear that g^{TM} is a Riemannian metric. We claim that ξ is the minus gradient vector field of f associated to g^{TM}, i.e., for any vector field X,

$$\langle \xi, X \rangle_{g^{TM}} = -X(f). \tag{4.0.11}$$

If $x \in \cup_p \overline{U'_p}$, then $\rho_1 = 0, \langle \xi, X \rangle_{g^{TM}} = \langle \xi, X \rangle_{g_0^{TM}}$ and

$$\langle \xi, X \rangle_{g^{TM}} = -X(f). \tag{4.0.12}$$

If $x \in M - \cup_p U_p$, then $\rho_0 = 0$, $\langle \xi, X \rangle_{g^{TM}} = \langle \xi, X \rangle_{g_1^{TM}}$ and

$$\langle \xi, X \rangle_{g^{TM}} = \begin{cases} 0 = -df(X) = -X(f), & \text{if } X \in N_x, \\ -\xi_x(f) = -X(f), & \text{if } X = \xi. \end{cases} \tag{4.0.13}$$

If $x \in \cup_p U_p - \cup_p \overline{U'_p}$, then

$$\begin{aligned} \langle \xi, X \rangle_{g^{TM}} =& \rho_0 \langle \xi, X \rangle_{g_0^{TM}} + \rho_1 \langle \xi, X \rangle_{g_1^{TM}} \\ =& -\rho_0 X(f) - \rho_1 X(f) = -X(f). \end{aligned} \tag{4.0.14}$$

From (4.0.12), (4.0.13) and (4.0.14), we get (4.0.11). The proof of Proposition 4.4 is complete. □

Set

$$\begin{aligned} C(f) =& \Big\{ p_{1_1}, \cdots, p_{1_{s_1}}, p_{2_1}, \cdots, p_{2_{s_2}}, \cdots, p_{r_1}, \cdots, p_{r_{s_r}} \Big\}, \\ f(p_{j_1}) =& f(p_{j_2}) = \cdots = f(p_{j_{s_j}}) = \overline{p}_j, \ j = 1, \cdots, r. \end{aligned} \tag{4.0.15}$$

Without loss of generality, we assume

$$\overline{p}_1 > \overline{p}_2 \cdots > \overline{p}_r. \tag{4.0.16}$$

Let ξ denote a minus gradient-like vector field of f. The following Lemma plays a crucial role in the proof of Theorem 4.1.

Lemma 4.5. *Given sufficient small $\varepsilon > 0, j$, there exist a minus gradient-like vector field $\overline{\xi} = \xi$ outside of $f^{-1}[\overline{p}_j + \varepsilon, \overline{p}_j + 2\varepsilon]$ and in the $\overline{\xi}$ system $W^u(p_{j_i}) \pitchfork W^s(p_k)$ for all i, k. $W^u(p_{j_i})$ in the $\overline{\xi}$ system has the obvious meaning.*

Proof. Let λ_{j_i} denote the index of $p_{j_i} \in C(f)$.
Set

$$S^u(p_{j_i}) = W^u(p_{j_i}) \cap f^{-1}(\overline{p}_j + \varepsilon), \ S^s(p_k) = W^s(p_k) \cap f^{-1}(\overline{p}_j + \varepsilon). \tag{4.0.17}$$

Then $S^u(p_{j_i})$ is a closed sphere in the hypersurface $f^{-1}(\overline{p}_j + 2\varepsilon)$ and $S^s(p_k)$ is a smooth submanifold in $f^{-1}(\overline{p}_j + 2\varepsilon)$. From the definition of the unstable manifolds,

$$S^u(p_{j_i}) \cap S^u(p_{j_l}) = \emptyset, \text{ if } i \neq l. \tag{4.0.18}$$

Since the unstable (stable) manifolds have transverse intersection with any level set of f in M. Then

$$W^u(p_{j_i}) \pitchfork W^s(p_k) \Longleftrightarrow S^u(p_{j_i}) \pitchfork S^s(p_k) \text{ in } f^{-1}(\overline{p}_j + \varepsilon). \tag{4.0.19}$$

We also have

$$\dim S^u(p_{j_i}) = \lambda_{j_i} - 1, \ \dim S^s(p_k) = n - 1 - \lambda_k. \tag{4.0.20}$$

We divide its proof into three steps.

Step 1. Let $\Phi_{j_i} : S^u(p_{j_i}) \times \mathbb{R}^{n-\lambda_{j_i}} \to U_{j_i} \subset f^{-1}(\overline{p}_j + \varepsilon)$ be a diffeomorphism onto a product neighborhood U_{j_i} of $S^u(p_{j_i})$ in $f^{-1}(\overline{p}_j + \varepsilon)$ such that $\Phi_{j_i}\big(S^u(p_{j_i}) \times 0\big) = S^u(p_{j_i})$. From (4.0.18), we also assume that

$$U_{j_i} \cap U_{j_l} = \emptyset, \text{ if } i \neq l. \tag{4.0.21}$$

Let g be the composition map:

$$S^s(p_k) \cap U_{j_i} \to U_{j_i} \to S^u(p_{j_i}) \times \mathbb{R}^{n-\lambda_{j_i}} \to \mathbb{R}^{n-\lambda_{j_i}}, \tag{4.0.22}$$

where the first map is the inclusion $i : S^s(p_k) \cap U_{j_i} \to U_{j_i}$, the second is the diffeomorphism $\Phi_{j_i}^{-1}$ and the last is the projection $\pi : S^u(p_{j_i}) \times \mathbb{R}^{n-\lambda_{j_i}} \to \mathbb{R}^{n-\lambda_{j_i}}$.

We claim that there exists a point $Z_{j_i} \in \mathbb{R}^{n-\lambda_{j_i}}$ such that

$$\Phi_{j_i}\big(S^u(p_{j_i}) \times Z_{j_i}\big) \pitchfork S^s(p_k), \quad \text{for all } k. \tag{4.0.23}$$

From (4.0.22), the manifold $\Phi_{j_i}\big(S^u(p_{j_i}) \times Z\big), Z \in \mathbb{R}^{n-\lambda_{j_i}}$ intersects $S^s(p_k)$ if and only if $Z \in g(S^s(p_k) \cap U_{j_i})$. Then we prove (4.0.23) as follows.

(1). If $\lambda_{j_i} \leqslant \lambda_k$, $g : S^s(p_k) \cap U_{j_i} \to \mathbb{R}^{n-\lambda_{j_i}}$ and $\dim\big(S^s(p_k) \cap U_{j_i}\big) \leqslant n-1-\lambda_k < n-\lambda_{j_i}$. By Sard's theorem, $g(S^s(x_2) \cap U_{j_i})$ has measure zero in $\mathbb{R}^{n-\lambda_{j_i}}$. Thus we may choose a point $Z_{j_i} \in \mathbb{R}^{n-\lambda_{j_i}} \backslash g(S^s(p_k) \cap U_{j_i})$ such that $\Phi_{j_i}\big(S^u(p_{j_i}) \times Z_{j_i}\big) \cap S^s(p_k) = \emptyset$.

(2). If $\lambda_j > \lambda_k$. We already know that $\Phi_{j_i}\big(S^u(p_j) \times Z\big) \cap S^s(p_k) = \emptyset$ if and only if Z does not belong to the image of g. Then we have the following discussion.

(2a). If $\Phi_{j_i}\big(S^u(p_j) \times Z\big) \cap S^s(p_k) = \emptyset$, then $Z \in \mathbb{R}^{n-\lambda_{j_i}} \backslash g(S^s(p_k) \cap U_{j_i})$. That is, there exists $Z_{j_i} \in \mathbb{R}^{n-\lambda_{j_i}} \backslash g(S^s(p_k) \cap U_{j_i})$ such that

$$\Phi_{j_i}\big(S^u(p_{j_i}) \times Z_{j_i}\big) \cap S^s(p_k) = \emptyset. \tag{4.0.24}$$

(2b). If $\Phi_{j_i}\big(S^u(p_j) \times Z\big) \cap S^s(p_k) \neq \emptyset$, then $Z \in g(S^s(p_k) \cap U_{j_i})$. Since Φ_{j_i} is a diffeomorphism, we get from (4.0.22) that

$$\begin{aligned} g \text{ is submersion} &\Leftrightarrow g_*\Big(\mathrm{T}\big(S^s(p_k) \cap U_{j_i}\big)\Big) = \mathbb{R}^{n-\lambda_{j_i}} \\ &\Leftrightarrow \pi_* \cdot \big(\Phi_{j_i}^{-1}\big)_*\Big(\mathrm{T}\big(S^s(p_k)\big)\Big) = \mathbb{R}^{n-\lambda_{j_i}} \\ &\Leftrightarrow \mathrm{T}\big(S^u(p_{j_i})\big) + \big(\Phi_{j_i}^{-1}\big)_*\Big(\mathrm{T}\big(S^s(p_k)\big)\Big) = \mathrm{T}\big(S^u(p_{j_i})\big) + \mathbb{R}^{n-\lambda_{j_i}} \\ &\Leftrightarrow \mathrm{T}\Big(\Phi_{j_i}\big(S^u(p_{j_i}) \times Z\big)\Big) + \mathrm{T}\big(S^s(p_k) = \mathrm{T}U_{j_i}. \end{aligned} \tag{4.0.25}$$

That is $\Phi_{j_i}\big(S^u(p_{j_i}) \times Z\big) \pitchfork S^s(p_k)$ if and only if g is submersion at $w \in S^s(p_k) \cap U_{j_i}$ for any $w \in g^{-1}(Z)$. By Sard's theorem, we can also choose a point $Z_{j_i} \in g\big(S^s(p_k) \cap U_{j_i}\big)$ such that $\Phi_{j_i}\big(S^u(p_{j_i}) \times Z_{j_i}\big) \pitchfork S^s(p_k)$.

Step 2. We will construct a diffeomorphism h of $f^{-1}(\overline{p}_j + \varepsilon)$ onto itself smoothly isotopic to the identity, such that $h\big(S^u(p_{j_i})\big)$ equals $\Phi_{j_i}\big(S^u(p_{j_i}) \times Z_{j_i}\big)$ for all i and thus has transverse intersection with $S^s(p_k)$.

We can easily construct a smooth vector field X_{j_i} on $\mathbb{R}^{n-\lambda_{j_i}}$ such that

$$X_{j_i}(Z) = \begin{cases} Z_{j_i}, & \text{if } |Z| \leqslant |Z_{j_i}|, \\ 0, & \text{if } |Z| \geqslant 2|Z_{j_i}|. \end{cases} \tag{4.0.26}$$

Let $\psi_{j_i,t}(Z)$ be the integral curve of X_{j_i}. Since $\operatorname{supp}(X_{j_i})$ is compact, $\psi_{j_i,t}(Z)$ is defined for all $t \in \mathbb{R}$. In fact, $\psi_{j_i,t}(Z)$ can be written as

$$\psi_{j_i,t}(Z) = \begin{cases} Z + tZ_{j_i}, & \text{if } |Z + tZ_{j_i}| \leqslant |Z_{j_i}|, \\ Z, & \text{if } |Z| \geqslant 2|Z_{j_i}|. \end{cases} \tag{4.0.27}$$

Then $\psi_{j_i,0}$ is the identity on $\mathbb{R}^{n-\lambda_{j_i}}$, $\psi_{j_i,1}$ is a diffeomorphism sending zero to Z_{j_i}, and $\psi_{j_i,t}$, $0 \leqslant t \leqslant 1$, gives a smooth isotopy from $\psi_{j_i,0}$ to $\psi_{j_i,1}$. Since this isotopy leaves all points fixed outside a bounded set in $\mathbb{R}^{n-\lambda_{j_i}}$ we can use it to define an isotopy

$$h_t : f^{-1}(\overline{p}_j + \varepsilon) \to f^{-1}(\overline{p}_j + \varepsilon) \tag{4.0.28}$$

by setting

$$h_t(w) = \begin{cases} \Phi_{j_i}\big(x, \psi_{j_i,t}(Z)\big), & \text{if } w = \Phi_{j_i}\big(x, Z\big) \in U_{j_i}, \\ w, & \text{if } w \in f^{-1}(\overline{p}_j + \varepsilon) \backslash \cup_{i=1}^{s_j} U_{j_i}. \end{cases} \tag{4.0.29}$$

Then $h = h_1$ is the desired diffeomorphism $f^{-1}(\overline{p}_j + \varepsilon) \to f^{-1}(\overline{p}_j + \varepsilon)$. Clearly,

$$h\big(S^u(p_{j_i})\big) = \Phi_{j_i}\big(S^u(p_{j_i}) \times Z_{j_i}\big). \tag{4.0.30}$$

Step 3. In this step, we denote $\overline{p}_j + \varepsilon$ and $\overline{p}_j + 2\varepsilon$ by a and b, respectively. We will alter the minus gradient vector field $\xi = -\nabla f$ (with respect to any given metric compatible with f) in $f^{-1}[a, b]$ and get another minus gradient-like vector field $\overline{\xi}$ such that in the $\overline{\xi}$ system,

$$\overline{S}^u(p_{j_i}) = h\big(S^u(p_{j_i})\big), \ \overline{S}^s(p_k) = S^s(p_k). \tag{4.0.31}$$

From (4.0.23), (4.0.30) and (4.0.31), we have

$$\overline{S}^u(p_{j_i}) \pitchfork \overline{S}^s(p_k), \quad \text{for all } i, k. \tag{4.0.32}$$

Let γ_t be the diffeomorphism generated by the vector field $\hat{\xi} = -\xi/\xi(f)$, that is

$$\frac{d\gamma_t(z)}{ds} = \hat{\xi}(\gamma_t(z)) \text{ and } \gamma_0(z) = z. \tag{4.0.33}$$

Then we have

$$\frac{df(\gamma_t(z))}{ds} = \langle \frac{d\gamma_t(z)}{ds}, \nabla f \rangle = -1. \tag{4.0.34}$$

Hence

$$f(\gamma_t(z)) = -t + f(z),\ \forall\ z \in f^{-1}[a,b]. \tag{4.0.35}$$

Now we define a diffeomorphism

$$\varphi : [a,b] \times f^{-1}(a) \to f^{-1}[a,b] \tag{4.0.36}$$

by setting

$$\varphi(t,x) = \gamma_{t-b}(x), \quad \text{for } t \in [a,b],\ x \in f^{-1}(a). \tag{4.0.37}$$

Then we have

$$f(\varphi(t,x)) = a + b - t,\ \varphi(b,x) = x, \forall\ x \in f^{-1}(a). \tag{4.0.38}$$

Define a diffeomorphism H of $[a,b] \times f^{-1}(a)$ onto itself by setting

$$H(t,x) = \big(t, h_t(x)\big), \tag{4.0.39}$$

where $h_t(x)$ is a smooth isotopy $[a,b] \times f^{-1}(a) \to f^{-1}(a)$ from the identity to h adjusted so that h_t is the identity for t near a and $h_t = h$ for t near b, i.e.,

$$h_t = \begin{cases} \text{Id}, & \text{if } t \ \text{ near } a, \\ h, & \text{if } t \ \text{ near } b. \end{cases} \tag{4.0.40}$$

Set

$$\xi^{'} = (\varphi \circ H \circ \varphi^{-1})_* \hat{\xi}. \tag{4.0.41}$$

We now claim that $\xi^{'}$ is a smooth vector field defined on $f^{-1}[a,b]$ which coincides with $\hat{\xi}$ near $f^{-1}(a)$ and $f^{-1}(b)$ and $\xi^{'}(f) \equiv -1$. In fact, we get from (4.0.37), (4.0.38) and (4.0.39) that

$$\varphi \circ H \circ \varphi^{-1} : f^{-1}[a,b] \to f^{-1}[a,b] \tag{4.0.42}$$

is given by

$$\varphi \circ H \circ \varphi^{-1}(z) = \gamma_{a-f(z)} \circ h_{a+b-f(z)} \circ \gamma_{f(z)-a}(z), \forall z \in f^{-1}[a,b]. \tag{4.0.43}$$

Set $w = \varphi \circ H \circ \varphi^{-1}(z) \in f^{-1}[a,b]$. Clearly, $f(w) = f(z)$. Then

$$\xi^{'}(f) = \big[(\varphi \circ H \circ \varphi^{-1})_{*,z} \hat{\xi}\big] f = \hat{\xi}_z \big[f(\varphi \circ H \circ \varphi^{-1})\big] = \hat{\xi}_z(f) \equiv -1. \tag{4.0.44}$$

From (4.0.39), (4.0.40) and (4.0.43), we find that when z is near $f^{-1}(b)$,

$$\varphi \circ H \circ \varphi^{-1}(z) = z,\ \xi^{'}_z = \hat{\xi}_z; \tag{4.0.45}$$

and when z is near $f^{-1}(a)$,

$$\begin{aligned}\xi'_w =& \frac{d}{dt}\left[\gamma_{t+a-f(z)}h_{t+a+b-f(z)}\gamma_{f(z)-t-a}\gamma_t(z)\right]\Big|_{t=0} \\ =& \frac{d}{dt}\left[\gamma_t(\gamma_{a-f(z)}h\gamma_{f(z)-a})\right]\Big|_{t=0} = \hat{\xi}_w. \end{aligned} \tag{4.0.46}$$

Next we define a smooth vector field $\overline{\xi}$ on M by

$$\overline{\xi}_z = \begin{cases} -\xi(f)\cdot\xi', & \text{if } z \in f^{-1}[a,b] \\ \xi, & \text{if elsewhere.} \end{cases} \tag{4.0.47}$$

Clearly, $\overline{\xi}(f) < 0$ on $f^{-1}[a,b]$ and $\overline{\xi}$ coincides with $\hat{\xi}$ elsewhere. Thus $\overline{\xi}$ is a minus gradient-like vector field.

We now verify that $\varphi(t, h_t(x))$ is an integral curve of $\overline{\xi}$ as follows. From (4.0.37),

$$\varphi\big(t, h_t(x)\big) = \gamma_{t-b}\big(h_t(x)\big). \tag{4.0.48}$$

On the other hand, fix $x \in f^{-1}(a)$, the integral curve of the vector field ξ':

$$\gamma_{t+a-f(x)} \circ h_{a+b+t-f(x)} \circ \gamma_{f(x)-a}(x) = \gamma_t \circ h_{b+t}(x). \tag{4.0.49}$$

Then we know that $\varphi(t, h_t(x))$ is the integral curve of the vector field ξ'. We can also see this point by direct computer as follows:

$$\xi' = (\varphi\circ H\circ\varphi^{-1})_*\hat{\xi} = (\varphi\circ H)_*\hat{\xi} = \varphi_*\Big[\frac{\partial}{\partial t} + \frac{\partial}{\partial t}h_t(x)\Big] = \frac{d}{dt}\varphi\big(t, h_t(x)\big). \tag{4.0.50}$$

Let $t(s)$ denote the solution of the ordinary partial problem

$$\frac{d}{ds}t(s) = -\big|\nabla f\big|^2_{\varphi(t(s),h_{t(s)}(x))}, \tag{4.0.51}$$

with the given initial data $t(0) = a$. Then $\varphi\big(t(s), h_{t(s)}(x)\big)$ is the integral curve of the vector field:

$$\begin{aligned}\frac{d}{ds}\varphi\big(t(s), h_{t(s)}(x)\big) =& \frac{d}{dt}\varphi\big(t(s), h_{t(s)}(x)\big)\cdot\frac{d}{ds}t(s) \\ =& \xi'_{\varphi(t(s),h_{t(s)}(x))}\cdot\frac{d}{ds}t(s) = \overline{\xi}_{\varphi(t(s),h_{t(s)}(x))}. \end{aligned} \tag{4.0.52}$$

Finally for each fixed $x \in f^{-1}(a)$, $\varphi(t, h_t(x))$ describes an integral curve of $\overline{\xi}$ from $\varphi(a, x)$ in $f^{-1}(b)$ to $\varphi\big(b, h(x)\big) = h(x)$ in $f^{-1}(a)$. It follows that

$$\overline{S}^u(p_{j_i}) = h\big(S^u(p_{j_i})\big),\ \overline{S}^s(p_k) = S^s(p_k). \tag{4.0.53}$$

□

Proof of Theorem 4.1. Introduce the following hypothesis:

$\mathcal{H}(q)$: There exist a gradient-like vector field $\xi_q = \xi$ in a neighborhood of the p_k such that in the ξ_q system, $W^u(p_{(r-j)_i}) \pitchfork W^s(p_k)$ for all $j \leqslant q, i = 1, \cdots, s_{r-j}$ and all k (We make the induction on the level set of f.) $\mathcal{H}(r)$ implies Theorem 4.1.

When $q = 0$, it is clear that $W^u(p_{r_i}) \pitchfork W^s(p_k)$ for all $i = 1, \cdots, s_r$ and all k since $W^u(p_{r_i}) = \{p_{r_i}\}, i = 1, \cdots, s_r$. We will show that $\mathcal{H}(q-1)$ implies $\mathcal{H}(q)$. Given ξ_{q-1} by $\mathcal{H}(q-1)$ we will construct ξ_q. Let $\varepsilon > 0$ be small enough and apply Lemma 4.5 to obtain a minus gradient-like vector field $\xi_q = \xi_{q-1}$ outside of $f^{-1}[\overline{p}_{r-q} + \varepsilon, \overline{p}_{r-q} + 2\varepsilon]$, and in the ξ_q system, for all k, $W^u(p_{(r-q)_i}) \pitchfork W^s(p_k), i = 1, \cdots, s_{r-q}$. But all $W^u(p_{j_i}) \pitchfork W^s(p_k), i = 1, \cdots, s_j$ for $j > r - q$ and all k since this is true in the ξ_{q-1} system, $\xi_q = \xi_{q-1}$ on $f^{-1}[\overline{p}_{r-q+1}, \overline{p}_r]$ and $W^u(p_{j_i}) \cap W^s(p_k) \subset f^{-1}[\overline{p}_{r-q+1}, \overline{p}_r], i = 1, \cdots, s_j$. This completes the proof of Theorem 4.1. $\square$

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Teilpublikationen liegen nicht vor.

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